Logic, Computability and Incompleteness

The Unprovability of Consistency

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Consistency Proofs

Consistency Proofs

At the turn of the 20th century, set-theoretic concepts became central to many areas of mathematics.

But these concepts give rise to paradoxes.

Frege's axiomatization of arithmetic turned out to be inconsistent, due to its endorsement of unrestricted comprehension.

ZFC was developed to avoid these paradoxes.

But does it?

Consistency Proofs

Hilbert's vision:

Provide finitary consistency proofs for ZFC and other foundational theories.

All we need to show is that $\perp$ is not provable from the axioms.

Gödel's Second Incompleteness Theorem (1931):

No sufficiently powerful and consistent axiomatic theory can prove its own consistency.

Therefore: If ZFC is consistent then no theory weaker than ZFC can prove the consistency of ZFC.

The Second Incompleteness Theorem

The Second Incompleteness Theorem

Theorem 8.3
All recursive functions and relations are representable in every extension of Q.

Let $\mathrm{Prf}_{\mathrm{PA}}$ be the relation that holds between *n* and *m* iff *n* codes a PA-proof of the sentence coded by *m*.

By Theorem 8.3, there is a formula $\textsc{prf}_{\mathrm{PA}}(x, y)$ such that for all natural numbers *n* and *m*,

(i) If $\mathrm{Prf}_{\mathrm{PA}}(n, m)$, then $\vdash_{\mathrm{PA}} \textsc{prf}_{\mathrm{PA}}(\overline{n}, \overline{m})$;

(ii) If not $\mathrm{Prf}_{\mathrm{PA}}(n, m)$, then $\vdash_{\mathrm{PA}} \neg\textsc{prf}_{\mathrm{PA}}(\overline{n}, \overline{m})$.

Define $\textsc{prov}_{\mathrm{PA}}(x)$ as $\exists x\, \textsc{prf}_{\mathrm{PA}}(y, x)$.

The Second Incompleteness Theorem

The Diagonal Lemma

For every PA-formula $A(x)$, there is a sentence G such that $\vdash_{\mathrm{PA}} G \leftrightarrow A(\ulcorner G \urcorner)$.

Apply the diagonal lemma to $\neg\mathsf{PROV}_{\mathrm{PA}}(x)$.

This gives us a sentence G such that $\vdash_{\mathrm{PA}} G \leftrightarrow \neg\mathsf{PROV}_{\mathrm{PA}}(\ulcorner G \urcorner)$.

The Second Incompleteness Theorem

Half of Gödel's First Incompleteness Theorem

If PA is consistent, then PA does not prove G.

Suppose PA proves G.

Then it proves $\neg\text{PROV}_{\mathrm{PA}}(\ulcorner G \urcorner)$ because it proves $G \leftrightarrow \neg\text{PROV}_{\mathrm{PA}}(\ulcorner G \urcorner)$.

But also, there is a proof of G, coded by some number n.

So $\mathrm{Prf}_{\mathrm{PA}}(n, \ulcorner G \urcorner)$ holds.

So PA proves $\text{PRF}_{\mathrm{PA}}(\overline{n}, \ulcorner G \urcorner)$,

So PA proves $\text{PROV}_{\mathrm{PA}}(\ulcorner G \urcorner)$.

So PA is inconsistent.

The Second Incompleteness Theorem

Half of Gödel's First Incompleteness Theorem

If PA is consistent, then PA does not prove G.

We can translate this into $\mathfrak{L}_A$:

$$\neg \mathsf{PROV}_{\mathrm{PA}}(\ulcorner \bot \urcorner) \to \neg \mathsf{PROV}_{\mathrm{PA}}(\ulcorner G \urcorner)$$

This sentence is provable in PA.

Suppose PA proves $\neg \mathsf{PROV}_{\mathrm{PA}}(\ulcorner \bot \urcorner)$.

Then it proves $\neg \mathsf{PROV}_{\mathrm{PA}}(\ulcorner G \urcorner)$.

So it proves G.

So it is inconsistent.

Provability Logic

$$\neg\mathsf{PROV}_{\mathrm{PA}}(\ulcorner\bot\urcorner) \to \neg\mathsf{PROV}_{\mathrm{PA}}(\ulcorner G\urcorner)$$

$$\neg\Box\bot \to \neg\Box G.$$

This is provable in any theory in which the box satisfies the Hilbert–Bernays–Löb conditions:

P1 *If* $\vdash_T A$, *then* $\vdash_T \Box A$.

P2 $\vdash_T \Box(A \to B) \to (\Box A \to \Box B)$.

P3 $\vdash_T \Box A \to \Box\Box A$.

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P2 $\vdash_T \Box(A \to B) \to (\Box A \to \Box B)$.

P3 $\vdash_T \Box A \to \Box\Box A$.

Löb's Theorem

If $\vdash_T \Box A \to A$, *then* $\vdash_T A$, *provided that the box satisfies P1–P3.*

Provability Logic

The full logic of provability:

Nec	*If* $\vdash_T A$, *then* $\vdash_T \Box A$.
K	$\vdash_T \Box(A \to B) \to (\Box A \to \Box B)$.
4	$\vdash_T \Box A \to \Box\Box A$.
GL	$\vdash_T \Box(\Box A \to A) \to \Box A$.

Exam

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In class there is a preview of the exam here.

PPLS Semester 1 Course Survey

Honours and Postgraduate

https://edin.ac/4ou0ttk

Opens: 17 November 2025

Closes: 5 December 2025