Logic, Computability and Incompleteness

Theories

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A (first-order) **theory** T is a set of sentences closed under entailment.

Notation: $T \vdash A$ (synonym of $A \in T$).

By completeness, $T \vdash A$ iff $T \models A$.

T is **axiomatized** by Γ iff $T = \{A : \Gamma \vdash A\}$.

Arithmetic

Language of first-order arithmetic

- Symbols: 0, s, $+$, $\times$.

All computable number-theoretic properties and relations are definable from $0, s, +, \times$.

- $x < y$ iff $\exists z\,(x + s(z) = y)$.
- t is prime iff
  $s(0) < t \wedge \forall y \big(\exists z\,(z \times y = t) \rightarrow (y = s(0) \vee y = t)\big)$.

Arithmetic

Robinson's Q:

Q1	$\forall x \forall y\, (s(x) = s(y) \rightarrow x = y)$
Q2	$\forall x\, 0 \neq s(x)$
Q3	$\forall x\, (x \neq 0 \rightarrow \exists y\, x = s(y))$
Q4	$\forall x\, (x + 0 = x)$
Q5	$\forall x \forall y\, (x + s(y) = s(x + y))$
Q6	$\forall x\, (x \times 0 = 0)$
Q7	$\forall x \forall y\, (x \times s(y) = (x \times y) + x)$

First-Order Peano Arithmetic (PA):

Q1	$\forall x \forall y\,(s(x) = s(y) \rightarrow x = y)$
Q2	$\forall x\, 0 \neq s(x)$
Ind	$A(0) \wedge \forall x\,(A(x) \rightarrow A(s(x))) \rightarrow \forall x\, A(x)$
Q4	$\forall x\,(x + 0 = x)$
Q5	$\forall x \forall y\,(x + s(y) = s(x + y))$
Q6	$\forall x\,(x \times 0 = 0)$
Q7	$\forall x \forall y\,(x \times s(y) = (x \times y) + x)$

Second-Order Peano Arithmetic (PA2):

Q1	$\forall x \forall y\, (s(x) = s(y) \rightarrow x = y)$
Q2	$\forall x\, 0 \neq s(x)$
Ind2	$\forall X(X(0) \wedge \forall y\, (X(y) \rightarrow X(s(y))) \rightarrow \forall y\, X(y))$
Q4	$\forall x\, (x + 0 = x)$
Q5	$\forall x \forall y\, (x + s(y) = s(x + y))$
Q6	$\forall x\, (x \times 0 = 0)$
Q7	$\forall x \forall y\, (x \times s(y) = (x \times y) + x)$

Nonstandard models of Q and PA (and $\mathrm{Th}(\mathfrak{A})$)

- Add a new constant c and sentences $c \neq 0$, $c \neq s(0)$, $c \neq s(s(0)), \ldots$
- Every finite subset has a model.
- By compactness: there is a model of Q/PA/$\mathrm{Th}(\mathfrak{A})$ where c is not any $s^n(0)$.

No first-order theory can exclude extra elements beyond the standard $0, 1, 2, \ldots$.

No nonstandard models of PA2

The second-order Induction axiom excludes nonstandard elements:

$$\forall X(X(0) \land \forall y\,(X(y) \rightarrow X(s(y))) \rightarrow \forall y\,X(y)$$

$$\text{Let } X(y) \iff \forall Z(Z(0) \land \forall x\,(Z(x) \rightarrow Z(s(x))) \rightarrow Z(y)$$

All models of PA2 are isomorphic.

Compactness fails in second-order logic.

Gödel's First Incompleteness Theorem

Every consistent, computably axiomatizable extension T of Q is incomplete: there is a sentence G such that neither $T \vdash G$ nor $T \vdash \neg G$.

- Q, PA, PA2 are all computably axiomatizable; $\mathrm{Th}(\mathfrak{A})$ is not.
- Q, PA, PA2 are all incomplete; $\mathrm{Th}(\mathfrak{A})$ is complete.
- But PA2 $\vDash A$ for all $A \in \mathrm{Th}(\mathfrak{A})$ (all models of PA2 are isomorphic to $\mathfrak{A}$).
- So PA2 $\vDash G$ or PA2 $\vDash \neg G$, even though PA2 $\nvdash G$ and PA2 $\nvdash \neg G$.
- So second-order logic has no sound and complete proof system.

Set Theory

The cumulative hierarchy V

- $V_0 = \varnothing$
- $V_1 = \mathcal{P}(V_0) = \{\varnothing\}$
- $V_2 = \mathcal{P}(V_1) = \{\varnothing, \{\varnothing\}\}$
- ...
- $V_\omega = \bigcup_{k<\omega} V_k$
- $V_{\omega+1} = \mathcal{P}(V_\omega)$
- ...
- $V_{\omega+\omega} = \bigcup_{n<\omega} V_{\omega+n}$
- ...
- At limit ordinals λ: $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$.

Language and contextual definitions

- Non-logical symbol: $\in$;

Define $\varnothing$ via:

- $A(\varnothing)$ iff $\exists x\big(\forall y \neg\, y \in x \wedge A(x)\big)$.

ZFC axioms:

Z1	*Extensionality:* $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \to x = y)$
Z2	*Separation (schema):* $\forall y \exists z \forall x(x \in z \leftrightarrow (x \in y \wedge A(x)))$
Z3	*Empty set:* $\exists x \forall y(y \notin x)$
Z4	*Union:* $\forall x \exists u \forall y(y \in u \leftrightarrow \exists z(z \in x \wedge y \in z))$
Z5	*Power set:* $\forall x \exists p \forall y(y \in p \leftrightarrow y \subseteq x)$
Z6	*Pairing:* $\forall x \forall y \exists z \forall v(v \in z \leftrightarrow (v = x \vee v = y))$
Z7	*Infinity:* $\exists x(\varnothing \in x \wedge \forall y(y \in x \to y \cup \{y\} \in x))$
Z8	*Foundation:* $\forall x(x \neq \varnothing \to \exists y(y \in x \wedge x \cap y = \varnothing))$
Z9	*Replacement (schema): if A defines a function whose domain is a set, then its image is a set.*
Z10	*Choice: from any set of nonempty sets, there is a choice function selecting one element from each.*

Sets and Numbers

Von Neumann ordinals

- $0 = \varnothing$,
- $1 = \{0\} = \{\varnothing\}$,
- $2 = \{0, 1\} = \{\varnothing, \{\varnothing\}\}$,
- ...

Successor: $s(x) = x \cup \{x\}$.

PA is **interpretable** in ZFC: every PA axiom translates to a theorem of ZFC.

Von Neumann ordinals

- $0 = \varnothing$,
- $1 = \{0\} = \{\varnothing\}$,
- $2 = \{0, 1\} = \{\varnothing, \{\varnothing\}\}$,
- ...
- $\omega = \{0, 1, 2, \ldots\}$
- $\omega + 1 = \{0, 1, 2, \ldots, \omega\}$
- $\omega + 2 = \{0, 1, 2, \ldots, \omega, \omega + 1\}$
- ...
- $\omega \cdot 2 = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots\}$
- ...

An ordinal is any set that is transitive and $\in$-ordered.

Cardinals as ordinals

The cardinality of a set X is the least ordinal equinumerous with X.

- $|\varnothing| = 0 = \varnothing$
- $|\{\varnothing\}| = 1 = \{\varnothing\}$
- $|\mathbb{N}| = \aleph_0 = \omega$
- $|\mathcal{P}(\mathbb{N})| > \omega$

Cardinals as ordinals

- $|\mathbb{N}| = \aleph_0 = \omega$
- $|\mathcal{P}(\mathbb{N})| > \omega$

The Continuum Hypothesis (CH): $|\mathcal{P}(\mathbb{N})| = \aleph_1$

- Gödel (1938): ZFC $\nvdash \neg$CH (if ZFC consistent).
- Cohen (1963): ZFC $\nvdash$ CH (if ZFC consistent).