Logic, Computability and Incompleteness

Completeness of first-order Logic

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Cardinalities

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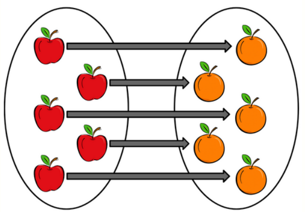
The Galileo-Hume-Cantor Principle: Two sets have the same size iff there is a one-to-one correspondence between their elements.



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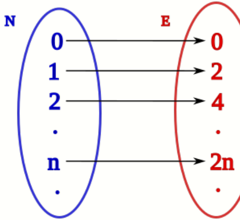

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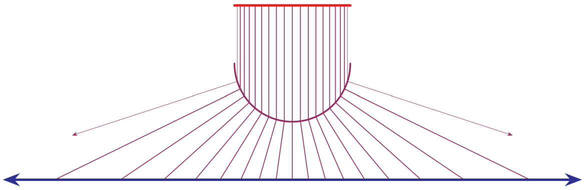
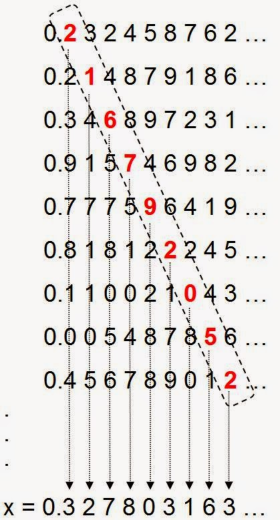


Cardinalities

Cantor's Theorem: There are uncountable sets.



Completeness of First-Order Logic

The Completeness Theorem: If $\Gamma \models A$, then $\Gamma \vdash A$.

Proof strategy:

- Assume $\Gamma \nvdash A$.
- Then $\Gamma \cup \{\neg A\}$ is consistent.
- Every consistent set extends to a maximal consistent set.
- From a maximal consistent set, we can read off a model that satisfies all sentences in the set.
- So there is a model that satisfies Γ and $\neg A$.
- So $\Gamma \not\models A$.

The Completeness Theorem: If $\Gamma \models A$, then $\Gamma \vdash A$.

Proof strategy:

- Assume $\Gamma \nvdash A$.
- Then $\Gamma \cup \{\neg A\}$ is consistent.
- Every consistent set extends to a Henkin set.
- From a Henkin set, we can read off a model that satisfies all sentences in the set.
- So there is a model that satisfies Γ and $\neg A$.
- So $\Gamma \not\models A$.

A Henkin set is maximal consistent and contains a "witness" $A(c)$ for each element $\exists x A(x)$.

Compactness

Compactness

The Compactness Theorem: If every finite subset of Γ is satisfiable, then Γ is satisfiable.

Typical application:

1. Take some theory T.
2. Add infinitely many sentences $S_1, S_2, S_3, \ldots$ to T of which any finite subset is obviously compatible with T.
3. Use compactness to infer that $T \cup \{S_1, S_2, S_3, \ldots\}$ has a model.

Compactness

The Compactness Theorem: If every finite subset of Γ is satisfiable, then Γ is satisfiable.

Example:

1. Take any consistent theory T of the natural numbers.
2. Add $0 < c$, $1 < c$, $2 < c$, ... to T.
3. By compactness, $T \cup \{0 < c, 1 < c, 2 < c, \ldots\}$ has a model.

Compactness

The Compactness Theorem: If every finite subset of Γ is satisfiable, then Γ is satisfiable.

Example:

1. Take any consistent theory T with a countably infinite model.
2. Add $c \neq d$ for uncountably many new constants c and d to T.
3. By compactness, T has an uncountable model.