Logic, Computability and Incompleteness

Propositional Logic

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26 September 2025

Overview

Overview

- Syntax vs. Semantics
- Proofs
- Models
- Soundness and Completeness

Syntax vs. Semantics

Syntax vs. Semantics

Syntax

- Definition of sentences (formation rules for $\mathcal{L}_0$)
- Proofs and derivations (axioms, rules like MP)
- Structural principles (Id, Mon, Cut)
- Deduction Theorem (DT)
- Negation principles (EFQ, DNE, RAA)
- Consistency: $\Gamma \nvdash_0 \bot$

Semantics

- Models (truth-assignments σ)
- Satisfaction $\sigma \Vdash A$
- Entailment $\Gamma \models A$
- Validity $\models A$
- Satisfiability

The propositional calculus

The propositional calculus

- Axioms (schemas):
  - A1: $A \to (B \to A)$
  - A2: $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
  - A3: $(\neg A \to \neg B) \to (B \to A)$
- Rule: Modus Ponens (MP): from A and $A \to B$, infer B.

Proofs are finite sequences.

$\vdash_0 A$.

The propositional calculus

- Deductions from premises: $\Gamma \vdash_0 A$ (each line is an axiom, an element of Γ, or follows by MP).
- Structural principles (immediate from the definition):
  - Id: $A \vdash_0 A$
  - Mon: If $\Gamma \vdash_0 A$ then $\Gamma, B \vdash_0 A$
  - Cut: If $\Gamma \vdash_0 A$ and $\Delta, A \vdash_0 B$ then $\Gamma, \Delta \vdash_0 B$
- Deduction Theorem (DT): If $\Gamma, A \vdash_0 B$ then $\Gamma \vdash_0 A \to B$.
- Negation principles EFQ, DNE, RAA are derivable.
- $\vdash_0$ is fully characterized by Id, Mon, Cut, MP, DT, EFQ, DNE, RAA.

A sequent calculus

- Id^+: $\Gamma, A \Rightarrow A$
- MP: from $\Gamma \Rightarrow A$ and $\Gamma \Rightarrow A \to B$, infer $\Gamma \Rightarrow B$
- DT: from $\Gamma, A \Rightarrow B$, infer $\Gamma \Rightarrow A \to B$
- RAA: from $\Gamma, A \Rightarrow B$ and $\Gamma, A \Rightarrow \neg B$, infer $\Gamma \Rightarrow \neg A$
- DNE: from $\Gamma \Rightarrow \neg\neg A$, infer $\Gamma \Rightarrow A$

Non-classical logics

- Intuitionistic logic drops DNE.
- Paraconsistent logics drop EFQ.
- Relevance logics restrict the structural rules so that the premises are "relevant" to the conclusions.
- Substructural logics (linear, affine, relevant, etc.) fine-tune specific structural rules.

Models

Models

Interpretations

A language isn't just a meaningless string of symbols.

'1+1=2' is a claim about the numbers 1 and 2 and the addition operation.

The **structure of the natural numbers** comprises

- a set of objects, $\mathbb{N} = \{0, 1, 2, \ldots\}$,
- operations (like addition and multiplication) on these objects,
- relations (like less-than) on these objects.

The **intended interpretation** of arithmetical sentences maps the symbols to these objects, operations, and relations.

Models

Interpretations

Whether a sentence is true normally depends on two things:

- what it means
- what is the case

A **model** provides just enough information about the two aspects to determine the truth-value of every sentences.

A model for propositional logic is a truth-value assignment σ to the sentence letters.

Models

Truth in a model ($\sigma \Vdash A$) is defined recursively:

- $\sigma \Vdash p$ iff $\sigma(p) = T$.
- $\sigma \Vdash \neg A$ iff not $\sigma \Vdash A$.
- $\sigma \Vdash A \to B$ iff not $\sigma \Vdash A$ or $\sigma \Vdash B$.

- $\Gamma \models A$ iff every σ with $\sigma \Vdash \Gamma$ also satisfies A.
- $\models A$ iff every model σ satisfies A.

Soundness and Completeness

Connection between syntax and semantics

- (Strong) Soundness: If $\Gamma \vdash_0 A$, then $\Gamma \models A$.
- (Strong) Completeness: If $\Gamma \models A$, then $\Gamma \vdash_0 A$.

- Weak Soundness: If $\vdash_0 A$, then $\models A$.
- Weak Completeness: If $\models A$, then $\vdash_0 A$.

Soundness and Completeness

Soundness

If $\Gamma \vdash_0 A$, then $\Gamma \models A$.

Proof:

- All axioms are valid.
- MP preserves truth in models.
- So: if all premises are true in all models, so is the conclusion.

Completeness

If $\Gamma \models A$, then $\Gamma \vdash_0 A$.

Strategy:

- Show that if $\Gamma \nvdash_0 A$, then $\Gamma \not\models A$.
- $\Gamma \nvdash_0 A$ iff $\Gamma \cup \{\neg A\}$ is consistent.
- $\Gamma \not\models A$ iff $\Gamma \cup \{\neg A\}$ is satisfiable.
- Need to show: every consistent set is satisfiable.

Completeness

If $\Gamma \models A$, then $\Gamma \vdash_0 A$.

Strategy:

- Need to show: every consistent set is satisfiable.
- Lindenbaum's Lemma: Every consistent set extends to a maximal consistent set.
- Truth Lemma: If Γ^+ is maximal consistent, the model σ_{Γ^+} that makes exactly the sentence letters in Γ^+ true satisfies all the sentences in Γ^+.