Logic, Computability and Incompleteness

First-order Logic

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First-order logic: Syntax

First-order logic: Syntax

Symbols:

- the connectives $\neg$ and $\rightarrow$
- the punctuation symbols '(', ')', and ','
- the identity symbol $=$
- the quantifier symbol $\forall$
- variables $x, y, z, \ldots$
- individual constants $a, b, c, \ldots$
- function symbols $f, g, h, \ldots$
- relation symbols $F, G, H, \ldots$

Terms:

- every variable and individual constant is a term.
- if $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol, then $f(t_1, \ldots, t_n)$ is a term.

First-order logic: Syntax

Atomic formulas:

- if t_1 and t_2 are terms, then $t_1 = t_2$ is an atomic formula.
- if $t_1, \ldots, t_n$ are terms and R is an n-ary relation symbol, then $Rt_1 \ldots t_n$ is an atomic formula.

Formulas:

- Every atomic formula is a formula.
- If A and B are formulas, then $\neg A$ and $(A \to B)$ are formulas.
- If A is a formula and x is a variable, then $\forall x A$ is a formula.

First-order logic: Syntax

Free and bound variables

- An occurrence of a variable x in a formula A is bound if it is inside a subformula of A of the form $\forall x B$.
- An occurrence of a variable x in a formula A is free if it is not bound.
- A formula is a sentence if it contains no free variables.

The first-order calculus

A1	$A \to (B \to A)$
A2	$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
A3	$(\neg A \to \neg B) \to (B \to A)$
A4	$\forall x A \to A(x/c)$
A5	$\forall x(A \to B) \to (A \to \forall x B),$ *if x is not free in A*
A6	$t = t$
A7	$t_1 = t_2 \to (A(x/t_1) \to A(x/t_2))$
MP	*From A and* $A \to B$ *one may infer B.*
Gen	*From A one may infer* $\forall x A(c/x)$.

The first-order calculus

Id	$A \vdash A$.
Mon	*If* $\Gamma \vdash A$ *then* $\Gamma, B \vdash A$.
Cut	*If* $\Gamma \vdash A$ *and* $\Delta, A \vdash B$ *then* $\Gamma, \Delta \vdash B$.
Taut	$\vdash A$ *whenever* A *is a truth-functional tautology.*
MP	*If* $\Gamma \vdash A$ *and* $\Gamma \vdash A \rightarrow B$ *then* $\Gamma \vdash B$.
DT	*If* $\Gamma, A \vdash B$ *then* $\Gamma \vdash A \rightarrow B$.
Gen	*If* $\Gamma \vdash A$ *then* $\Gamma \vdash \forall x A(c/x)$.
UI	*If* $\Gamma \vdash \forall x A$ *then* $\Gamma \vdash A(x/c)$.

First-order logic: Semantics

Models

Informally, a model describes a scenario together with an interpretation of the non-logical vocabulary.

- D is the intended domain of discourse.
- I maps each constant to an element of D, each function symbol to a function on D, and each non-logical predicate to a set of tuples from D.

First-order logic: Semantics

Satisfaction

$\mathfrak{M} \vDash A$: A is true in $\mathfrak{M}$.

Defined recursively.

First-order logic: Semantics

An example of a model $\mathfrak{M}$:

- Domain $D = \{0, 1\}$.
- $I(a) = 0, \quad I(b) = 1$.
- $I(f)$ is the function with $f(0) = 1,\ f(1) = 1$.
- $I(F) = \{1\}$.
- $I(R) = \{\langle 0, 1\rangle\}$.

Check:

(a) $\mathfrak{M} \models a = b$?
(b) $\mathfrak{M} \models f(a) = b$?
(c) $\mathfrak{M} \models \forall x Fx$?
(d) $\mathfrak{M} \models \forall x F f(x)$?
(e) $\mathfrak{M} \models \exists x R(x, f(x))$?
(f) $\mathfrak{M} \models \exists x (x = f(x) \rightarrow Fx)$?

Soundness

Soundness

If $\vdash A$, then $\models A$.

Proof:

- All axioms are valid.
- MP and Gen preserve validity.
- So: Anything that is provable is valid.

Corollary: If $\Gamma \vdash A$, then $\Gamma \models A$.

Second-order logic

Second-order logic

Syntax

- Second-order variables $X, Y, Z, \ldots$.
- If $t_1, \ldots, t_n$ are terms and X is an n-ary second-order variable, then $Xt_1 \ldots t_n$ is an atomic formula.
- If A is a formula and X is a second-order variable, then $\forall XA$ is a formula.

Semantics

- $\forall X A$ is true in $\mathfrak{M}$ if $A(X/P)$ is true in every model $\mathfrak{M}'$ that differs from $\mathfrak{M}$ at most in what it assigns to P.

In effect, the second-order variables range over all subsets of the domain.

Semantics

- $\forall XA$ is true in $\mathfrak{M}$ if $A(X/P)$ is true in every model $\mathfrak{M}'$ that differs from $\mathfrak{M}$ at most in what it assigns to P.

We can define '=':

$\mathfrak{M} \vDash s = t$ iff $\mathfrak{M} \vDash \forall X(Xs \leftrightarrow Xt)$.

Second-Order Axioms

A4	$\forall x A \rightarrow A(x/c)$
A4^2	$\forall X A \rightarrow A(X/P)$
A5	$\forall x(A \rightarrow B) \rightarrow (A \rightarrow \forall x B),$ *if x is not free in A*
A5^2	$\forall X(A \rightarrow B) \rightarrow (A \rightarrow \forall X B),$ *if X is not free in A*
Gen	*From A one may infer* $\forall x A(c/x)$.
Gen^2	*From A one may infer* $\forall X A(P/X)$.
Comp	$\exists X \forall y(Xy \leftrightarrow A(y))$.

This calculus is sound. It is not complete.