

Logic 2: Modal Logic

Lectures 21 and 22

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Semester 2 - Honours and Postgraduate

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Intermediate Logic

What we've covered:

- Translating from English into a formal language
- Reasoning in a formal language
- Different styles of proof
- The concept of a model
- Soundness and completeness (and compactness)
- Three-valued logic, supervaluation, 2D semantics, ...
- Sets, relations, orders, ...

What we've covered:

- Epistemic logic
- Deontic logic
- Temporal logic
- Conditionals

Review: Language

Formal logic studies artificial languages.

This is mainly to bypass some of the complexities of natural language.

We have looked at formal languages that extend the language of classical propositional or predicate logic by new sentence operators:

- \Box, \Diamond (K, M, B, O, P, G, F)
- \Box_1, \Box_2, \dots
- H, P
- $O(\cdot/\cdot), \neg, \Box \rightarrow$

We have used the extended language to formalise reasoning with **non-truth-functional** concepts.

- knowledge
- belief
- provability
- obligation and permission
- what will or was the case
- what could have been the case
- what would have been the case if so-and-so had been the case
- ...

Review: Language

Every modal operator has a **dual**:

$\Diamond A$ is equivalent to $\neg \Box \neg A$

$\Box A$ is equivalent to $\neg \Diamond \neg A$

Heuristics for translating from English into the language of modal logic:

- First paraphrase the original English sentence in such a way that all relevant modal and quantificational elements are turned into sentence operators: 'it is necessary that', 'it is possible that', 'it is required that', 'everything is such that', ...
- Avoid 'if-then' constructions in your paraphrase.
- Make sure your sentence letters stand for complete sentences that don't contain any relevant logical expressions.
- Check if you can think of a scenario in which your translation and the original sentence have different truth-values. Try edge cases!
- Avoid $A \rightarrow \Box B$.
- Avoid $\Diamond(A \rightarrow B)$.

Possible exam question:

Translate the following sentences, as well as possible, into a suitable modal language. (The resources of modal predicate logic are only needed for d.)

- (a) You can keep your shoes on.
- (b) I will never go to Italy.
- (c) If I fail this exam I have to do a resit.
- (d) Students who fail the exam can still pass the course.

Review: Proofs

A *proof method* is a rigorous method for checking whether a conclusion follows from some premises, or whether a sentence is logically true.

The oldest proof method is the *axiomatic method*.

An **axiomatic proof** of a sentence A is a list of sentences each of which is either an axiom (of the relevant system) or follows from earlier items by one of the rules (of the system).

An axiomatic calculus for the modal propositional logic K:

$$(A1) \quad A \rightarrow (B \rightarrow A)$$

$$(A2) \quad (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$$

$$(A3) \quad (\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)$$

$$(Dual) \quad \neg \Diamond A \leftrightarrow \Box \neg A$$

$$(K) \quad \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$$

(MP) If A and $A \rightarrow B$ occur on a proof, you may append B .

(Nec) If A occurs on a proof, you may append $\Box A$.

An alternative to the axiomatic method is the tableau method, or tree method.

In a **tree proof** for a sentence A , we start with a node $\neg A(w)$.

Then we expand the nodes on the tree in accordance with the tree rules of the relevant system.

If the tree closes, the target sentence A is valid.

Possible exam question:

Use the tree method to investigate the following claims. If a claim is false, give a countermodel in addition to the tree.

(a) $\models_K \Diamond p \rightarrow \Diamond(p \vee q)$

(b) $\models_{CK4} \Box \forall x \Box (Fx \rightarrow \Box Fx)$

If two proof methods allow proving the very same sentences, they are considered equivalent.

The set of sentences that can be proved with a certain method is called a **logic** or **system**.

There are infinitely many modal logics.

Possible exam questions:

1. Explain why everything that is provable in the axiomatic calculus for K is provable in the axiomatic calculus for S5.
2. Suppose we add the schema A to the axiomatic calculus for K. Is the resulting logic stronger or weaker than S5? Explain.
3. Consider the system that extends system K by all instances of the (T)-schema. Is the result the logic T? Explain.

Review: Models

Informally, A logically entails B (for short, $A \models B$) iff there is no conceivable scenario in which A is true and B is false, under any interpretation of the non-logical vocabulary.

We can represent a conceivable scenario and an interpretation of the non-logical vocabulary by a **model**.

A entails B iff $A \rightarrow B$ is valid.

To clarify the logic of modal operators, we often use **possible-worlds models**.

Guiding intuition:

- p is possible iff p is true at some (relevantly) possible world.
- p is necessary iff p is true at every (relevantly) possible world.

A **Kripke model** for modal propositional logic consists of

- a set of “worlds” W ,
- a binary “accessibility” relation R on W , and
- an interpretation function V that assigns to each sentence letter of \mathcal{L}_M a subset of W .

Kripke semantics for modal propositional logic specifies, for any world w in any Kripke model M , and any \mathcal{L}_M -sentence A , whether A is true at w in M .

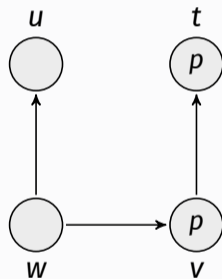
The **accessibility relation** in a Kripke model represents different things, depending on the application.

- wRv iff v is compatible with the laws of physics at w .
- wRv iff v is compatible with the knowledge at w .
- wRv iff v is compatible with the norms at w .
- wRv iff v is compatible with the essence of things.
- wRv iff v is later than w .
- ...

Possible exam question:

In the Kripke model on the right, p is true at v and t and false at w and u .

1. At which worlds in the model is $\Box p$ true?
2. For each world in the model, find an \mathcal{L}_M -sentence that is true only at that world.



Different interpretations of accessibility come with different formal constraints on Kripke models.

- R is **reflexive** if every world is accessible from itself.
- R is **serial** if every world can access some world.
- R is **universal** if every world can access every world.
- R is **transitive** if whenever wRv and vRu then wRu .
- R is **euclidean** if whenever wRv and wRu then vRu .
- ...

Imposing such a constraint on a Kripke model often changes which sentences are valid.

The set of sentences that are valid in a certain class of models is called a **logic** or **system**.

- A sentence is **K-valid** if it is true at all worlds in all Kripke models.
- A sentence is **T-valid** if it is true at all worlds in all Kripke models in which R is reflexive.
- A sentence is **S5-valid** if it is true at all worlds in all Kripke models in which R is universal (or: an equivalence relation)
- A sentence is **S4-valid** if it is true at all worlds in all Kripke models in which R is reflexive and transitive.

There are infinitely many modal logics.

The **system K** is (1) the set of all sentences that are true at all worlds in all Kripke models, and (2) the set of all sentences that can be proved with the rules of a certain axiomatic calculus or tree method.

That (1) and (2) define the same system is established by soundness and completeness of the relevant proof method.

A method is **sound** if anything that's provable with the method is valid.

A method is **complete** if anything that's valid is provable with the method.

Possible exam question:

Consider the follow interpretations of the box. For each of them, explain if we can use Kripke semantics for the relevant models. If we can, also explain what constraints we should impose on the accessibility relation.

- (a) It is true that
- (b) It is false that
- (c) I once believed that

Terminology:

- A formal sentence can be **true at** a world in a model. It can't be **valid at** a world.
- A sentence can be **valid in** a class of Kripke models. It can't be **true in** such a class.
- A system of modal logic does not have an accessibility relation.
- A system of modal logic does not have any rules.
- A system of modal logic does not contain any schemas.

A **variable-domain Kripke model** for modal predicate logic consists of

1. a non-empty set W (the “worlds”),
2. a binary (“accessibility”) relation R on W ,
3. for each world w , a non-empty set D_w (of “individuals”), and
4. an interpretation function V that assigns
 - to each name a member of some domain D_w , and
 - to each n -place predicate and world w a set of n -tuples from D_w .

In a **constant-domain Kripke model**, all worlds are associated with the same domain D .

Possible exam questions:

1. Explain why every K-valid \mathcal{L}_M -sentence is S4-valid.
2. Show that for any sentences A and B , if $\models_K A \rightarrow B$, then $\models_K \Diamond A \rightarrow \Diamond B$.
3. Give a constant-domain countermodel for $\exists x \Box Fx$ with a universal accessibility relation.
4. Show that there is no \mathcal{L}_M -sentence that is valid on all and only the finite models.
5. Are all instances of $\Box(\Box A \rightarrow A) \rightarrow (\Diamond \Box A \rightarrow \Box A)$ valid in the class of Kripke models in which R is a (strict) linear order? If yes, explain briefly. If no, give a counterexample.

Review: Frames and correspondence

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A Kripke model has three parts: W, R, V .

When we define validity (or entailment) in terms of a class of Kripke models, we never put constraints on V .

We effectively define validity with respect to a class of Kripke frames.

A **frame** is a model without an interpretation function.

A sentence is **valid on a frame** iff it is true at all worlds in all models based on that frame.

Review: Frames and correspondence

Constraints on the accessibility relation often **correspond** to modal schemas, in the sense that all instances of the schema are valid on a frame iff the frame satisfies the constraint.

<i>Schema</i>	<i>Corresponding Frame Condition</i>
T $\Box A \rightarrow A$	R is reflexive: every world is accessible from itself
D $\Box A \rightarrow \Diamond A$	R is serial: every world can access some world
B $A \rightarrow \Box \Diamond A$	R is symmetric: whenever wRv then vRw
4 $\Box A \rightarrow \Box \Box A$	R is transitive: whenever wRv and vRu , then wRu
5 $\Diamond A \rightarrow \Box \Diamond A$	R is euclidean: whenever wRv and wRu , then vRu such that vRt and uRt

Review: Frames and correspondence

Sentences or schemas that contain more than one type of box or diamond are called **interaction principles**.

Interaction principles often correspond to joint constraints on different accessibility relations.

<i>Schema</i>	<i>Corresponding Frame Condition</i>
$\Box_1 A \rightarrow \Box_2 A$	Whenever wR_2v then wR_1v
$\Diamond_1 A \rightarrow \Box_2 \Diamond_1 A$	Whenever wR_1v and wR_2u then uR_1v .

Possible exam questions:

1. Find a frame condition that corresponds to $\Diamond A \rightarrow \Box A$.
2. Find a frame condition that corresponds to $A \rightarrow \Box A$.
3. What frame condition does the schema $(G(GA \rightarrow A) \rightarrow (FGA \rightarrow GA)) \wedge (H(HA \rightarrow A) \rightarrow (PHA \rightarrow HA))$ in temporal logic correspond to?

Review: Extra parameters

Review: Extra parameters

In Ockhamist semantics, truth is defined relative to an extra parameter that does not reflect an aspect of a conceivable scenario: a history.

We also need an extra (time or world) parameter if we want a 'Now' operator (or the modal analogue, 'Actually').

In first-order logic, we have an extra parameter g for assignment functions.

The method of supervaluation can be used to reduce extra parameters.

$M, t \models A$ iff $M, t, h \models A$ for all histories h through t .

Review: Ordering models

Operators for obligation and permission arguably quantify over the best of the circumstantially accessible worlds.

To make this explicit, models of deontic logic need

- a circumstantial accessibility relation, and
- an ordering of worlds as better and worse.

Review: Ordering models

With deontic ordering models, we can also define a conditional obligation operator:

$O(p/q)$ is true at w iff p is true at the best of the circumstantially accessible worlds at which q is true.

Formally, $A \square \rightarrow B$ has the same interpretation. The ordering is interpreted not as “better relative to w ” but as “more similar to w ”.

Review: Conditionals

Review: Conditionals

Some think subjunctive conditionals in English should be formalized in terms of $\Box \rightarrow$.

Others think they are strict conditionals $A \rightarrow B$.

Some think indicative conditionals in English should be formalized in terms of $\Box \rightarrow$.

Others think they are strict conditionals $A \rightarrow B$.

Others think they are material conditionals $A \rightarrow B$.

Others think indicative conditionals do not express propositions at all.

Beyond Logic 2

Other topics in modal logic:

- Intuitionistic logic, dynamic logic, game logic, description logic, arrow logic, ...
- Dynamic epistemic logic, epistemic-deontic logic, ...
- Hybrid logics, many-dimensional logics, description logic, ...
- Model operations: bisimilarity, p-morphisms, bulldozing, ...

Adding expressive resources to classical logic:

- Modal logic: $\Box p$
- Predicate logic with function terms: $\exists x(f(x) = b)$
- Second-order logic: $\exists F\forall xFx$
- Plural logic: $\exists xxFxx$
- ...

Seriously nonclassical logics:

- Free logic: $\not\models \exists x(x = a)$
- Many-valued logics: $A, B \not\models A \wedge B$
- Intuitionistic logic: $\neg\neg p \not\models p$
- Substructural logics

Limitative results for formal theories:

- Gödel's first incompleteness theorem
- Gödel's second incompleteness theorem
- Tarski's theorem
- Löb's theorem
- ...

Thanks!