

Logic 2: Modal Logic

Lecture 22

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Review: Models

Informally, A logically entails B (for short, $A \models B$) iff there is no conceivable scenario in which A is true and B is false, under any interpretation of the non-logical vocabulary.

We can represent a conceivable scenario and an interpretation of the non-logical vocabulary by a **model**.

A entails B iff $A \rightarrow B$ is valid.

To clarify the logic of modal operators, we often use **possible-worlds models**.

Guiding intuition:

- p is possible iff p is true at some (relevantly) possible world.
- p is necessary iff p is true at every (relevantly) possible world.

A **Kripke model** for modal propositional logic consists of

- a set of “worlds” W ,
- a binary “accessibility” relation R on W , and
- an interpretation function V that assigns to each sentence letter of \mathcal{L}_M a subset of W .

Kripke semantics for modal propositional logic specifies, for any world w in any Kripke model M , and any \mathcal{L}_M -sentence A , whether A is true at w in M .

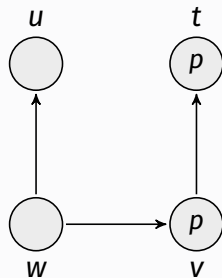
The **accessibility relation** in a Kripke model represents different things, depending on the application.

- wRv iff v is compatible with the laws of physics at w .
- wRv iff v is compatible with the knowledge at w .
- wRv iff v is compatible with the norms at w .
- wRv iff v is compatible with the essence of things.
- wRv iff v is later than w .
- ...

Possible exam question:

In the Kripke model on the right, p is true at v and t and false at w and u .

1. At which worlds in the model is $\Box p$ true?
2. For each world in the model, find an \mathcal{L}_M -sentence that is true only at that world.



Different interpretations of accessibility come with different formal constraints on Kripke models.

- R is **reflexive** if every world is accessible from itself.
- R is **serial** if every world can access some world.
- R is **universal** if every world can access every world.
- R is **transitive** if whenever wRv and vRu then wRu .
- R is **euclidean** if whenever wRv and wRu then vRu .
- ...

Imposing such a constraint on a Kripke model often changes which sentences are valid.

The set of sentences that are valid in a certain class of models is called a **logic** or **system**.

- A sentence is **K-valid** if it is true at all worlds in all Kripke models.
- A sentence is **T-valid** if it is true at all worlds in all Kripke models in which R is reflexive.
- A sentence is **S5-valid** if it is true at all worlds in all Kripke models in which R is universal (or: an equivalence relation)
- A sentence is **S4-valid** if it is true at all worlds in all Kripke models in which R is reflexive and transitive.

There are infinitely many modal logics.

The **system K** is (1) the set of all sentences that are true at all worlds in all Kripke models, and (2) the set of all sentences that can be proved with the rules of a certain axiomatic calculus or tree method.

That (1) and (2) define the same system is established by soundness and completeness of the relevant proof method.

A method is **sound** if anything that's provable with the method is valid.

A method is **complete** if anything that's valid is provable with the method.

Possible exam question:

Consider the follow interpretations of the box. For each of them, explain if we can use Kripke semantics for the relevant models. If we can, also explain what constraints we should impose on the accessibility relation.

- (a) It is true that
- (b) It is false that
- (c) I once believed that

Terminology:

- A formal sentence can be **true at** a world in a model. It can't be **valid at** a world.
- A sentence can be **valid in** a class of Kripke models. It can't be **true in** such a class.
- A system of modal logic does not have an accessibility relation.
- A system of modal logic does not have any rules.
- A system of modal logic does not contain any schemas.

A **variable-domain Kripke model** for modal predicate logic consists of

1. a non-empty set W (the “worlds”),
2. a binary (“accessibility”) relation R on W ,
3. for each world w , a non-empty set D_w (of “individuals”), and
4. an interpretation function V that assigns
 - to each name a member of some domain D_w , and
 - to each n -place predicate and world w a set of n -tuples from D_w .

In a **constant-domain Kripke model**, all worlds are associated with the same domain D .

Possible exam questions:

1. Give an example of an \mathcal{L}_M -sentences that is K-valid but that is not an instance of the (K)-schema $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$.
2. Explain why every K-valid \mathcal{L}_M -sentence is S4-valid.
3. Show that for any sentences A and B , if $\models_K A \rightarrow B$, then $\models_K \Diamond A \rightarrow \Diamond B$.
4. Give a constant-domain countermodel for $\exists x \Box Fx$ with a universal accessibility relation.

Review: Frames and correspondence

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A Kripke model has three parts: W, R, V .

When we define validity (or entailment) in terms of a class of Kripke models, we never put constraints on V .

We effectively define validity with respect to a class of Kripke frames.

A **frame** is a model without an interpretation function.

A sentence is **valid on a frame** iff it is true at all worlds in all models based on that frame.

Review: Frames and correspondence

Constraints on the accessibility relation often **correspond** to modal schemas, in the sense that all instances of the schema are valid on a frame iff the frame satisfies the constraint.

<i>Schema</i>	<i>Corresponding Frame Condition</i>
T $\Box A \rightarrow A$	R is reflexive: every world is accessible from itself
D $\Box A \rightarrow \Diamond A$	R is serial: every world can access some world
B $A \rightarrow \Box \Diamond A$	R is symmetric: whenever wRv then vRw
4 $\Box A \rightarrow \Box \Box A$	R is transitive: whenever wRv and vRu , then wRu
5 $\Diamond A \rightarrow \Box \Diamond A$	R is euclidean: whenever wRv and wRu , then vRu such that vRt and uRt

Review: Frames and correspondence

Sentences or schemas that contain more than one type of box or diamond are called **interaction principles**.

Interaction principles often correspond to joint constraints on different accessibility relations.

Review: Extra parameters

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In Ockhamist semantics, truth is defined relative to an extra parameter that does not reflect an aspect of a conceivable scenario: a history.

We also need an extra (time or world) parameter if we want a 'Now' operator (or the modal analogue, 'Actually').

In first-order logic, we have an extra parameter g for assignment functions.

The method of supervaluation can be used to reduce extra parameters.

$M, t \models A$ iff $M, t, h \models A$ for all histories h through t .