Logic 2: Modal Logic

Lecture 21

Wolfgang Schwarz

03 April 2023

University of Edinburgh

Review: Language

Review: Language

Formal logic studies artificial languages.

This is mainly to bypass some of the complexities of natural language.

We have looked at formal languages that extend the language of classical propositional or predicate logic by new sentence operators

- $\Box, \Diamond$ (K, M, B, O, P, G, F)
- $\Box_1, \Box_2, \ldots$
- H, P
- $O(\cdot/\cdot)$, $\dashv\!\!3$, $\Box\!\!\rightarrow$

Review: Language

We have used the extended language to formalise reasoning with **non-truth-functional** concepts.

- knowledge
- belief
- provability
- obligation and permission
- what will or was the case
- what could have been the case
- what would have been the case if so-and-so had been the case
- ...

Review: Language

Every modal operator has a **dual**:

$\Diamond A$ is equivalent to $\neg\Box\neg A$

$\Box A$ is equivalent to $\neg\Diamond\neg A$

Review: Language

Heuristics for **translating from English into the language of modal logic**:

- First paraphrase the original English sentence in such a way that all relevant modal and quantificational elements are turned into sentence operators: 'it is necessary that', 'it is possible that', 'it is required that', 'everything is such that', ...
- Avoid 'if–then' constructions in your paraphrase.
- Make sure your sentence letters stand for complete sentences that don't contain any relevant logical expressions.
- Check if you can think of a scenario in which your translation and the original sentence have different truth-values. Try edge cases!
- Avoid $A \rightarrow \Box B$.
- Avoid $\Diamond(A \rightarrow B)$.

Review: Language

Possible exam question:

Translate the following sentences, as well as possible, into a suitable modal language. (The resources of modal predicate logic are only needed for d.)

(a) You can keep your shoes on.
(b) I will never go to Italy.
(c) If I fail this exam I have to do a resit.
(d) Students who fail the exam can still pass the course.

Review: Proofs

Review: Proofs

Having introduced a language, we can formalise reasoning about the relevant subject matter.

$\Diamond r$

$\Box(r \rightarrow w)$

$\Diamond w$

Review: Proofs

A *proof method* is a rigorous method for checking whether a conclusion follows from some premises, or whether a sentence is logically true.

The oldest proof method is the *axiomatic method*.

An **axiomatic proof** of a sentence A is a list of sentences each of which is either an axiom (of the relevant system) or follows from earlier items by one of the rules (of the system).

Review: Proofs

An axiomatic calculus for the modal propositional logic K:

(**A1**) $A \to (B \to A)$

(**A2**) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(**A3**) $(\neg A \to \neg B) \to (B \to A)$

(**Dual**) $\neg\Diamond A \leftrightarrow \Box\neg A$

(**K**) $\Box(A \to B) \to (\Box A \to \Box B)$

(**MP**) If A and $A \to B$ occur on a proof, you may append B.

(**Nec**) If A occurs on a proof, you may append $\Box A$.

An axiomatic calculus for the modal predicate logic CK:

(**A1**) $A \to (B \to A)$

(**A2**) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(**A3**) $(\neg A \to \neg B) \to (B \to A)$

(**Dual**) $\neg\Diamond A \leftrightarrow \Box\neg A$

(**K**) $\Box(A \to B) \to (\Box A \to \Box B)$

($\forall\exists$) $\neg\exists\chi A \leftrightarrow \forall\chi\neg A$

(**UI**) $\forall\chi A \to A[\eta/\chi]$

(**DI**) $\forall\chi(A \to B) \to (A \to \forall\chi B)$, if χ is not free in A

(**BF**) $\forall x\Box A \to \Box\forall x A$

(**ND**) $\forall x\forall y(x \neq y \to \Box x \neq y)$

(**MP**) If A and $A \to B$ occur on a proof, you may append B.

(**Nec**) If A occurs on a proof, you may append $\Box A$.

(**Gen**) If A occurs on a proof, you may append $\forall\chi A[\chi/\eta]$.

Review: Proofs

If two proof methods allow proving the very same sentences, they are considered equivalent.

The set of sentences that can be proved with a certain method is called a **logic** or **system**.

By adding axiom schemas to the axioms of system K (or CK) we get stronger logics.

- System T results from adding $\Box A \rightarrow A$ to the axioms and rules of K.
- System S4 results from adding $\Box A \rightarrow A$ and $\Box A \rightarrow \Box\Box A$.
- System S5 results from adding $\Box A \rightarrow A$ and $\Diamond A \rightarrow \Box\Diamond A$.
- ...

There are infinitely many modal logics.

Review: Proofs

Possible exam questions:

1. Explain why everything that is provable in the axiomatic calculus for S5 is provable in the axiomatic calculus for K.
2. Suppose we add the schema *A* to the axiomatic calculus for K. Is the resulting logic stronger or weaker than S5? Explain.
3. Consider the system that extends system K by all instances of the (T)-schema. Is the result the logic T? Explain.

Review: Proofs

An alternative to the axiomatic method is the tableau method, or tree method.

In a **tree proof** for a sentence A, we start with a node $\neg A\,(w)$.

Then we expand the nodes on the tree in accordance with the tree rules of the relevant system.

If the tree closes, the target sentence A is valid.

Review: Proofs

Possible exam question:

Use the tree method to investigate the following claims. If a claim is false, give a countermodel in addition to the tree.

(a) $\models_K \Diamond p \to \Diamond(p \vee q)$

(b) $\models_{CK4} \Diamond \forall x \Box (Fx \to \Box Fx)$