Logic 2: Modal Logic

Lecture 20

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Review

Review

$$(\textbf{BF}) \quad \forall x \Box A \rightarrow \Box \forall x A$$

$$(\textbf{CBF}) \quad \Box \forall x A \rightarrow \forall x \Box A$$

(BF) corresponds to non-increasing domains.

(CBF) corresponds to non-decreasing domains.

If we combine the **tree rules** for K with those for classical predicate logic, we can prove all instances of (BF) and (CBF).

If we combine the **axiomatic rules** for K with those for classical predicate logic, we can prove all instances of (CBF) but not of (BF).

$$(\textbf{BF})\ \ \forall x\Box A \rightarrow \Box\forall x A$$

$$(\textbf{CBF})\ \ \Box\forall x A \rightarrow \forall x\Box A$$

(BF) corresponds to non-increasing domains.

(CBF) corresponds to non-decreasing domains.

To reason about variable domains, we need to change the underlying predicate logic to a free logic.

The necessity of identity and distinctness

The necessity of identity and distinctness

The following sentences are CK-valid and VK-valid:

$$(\textbf{NI})\ \ a = b \rightarrow \Box a = b$$

$$(\textbf{ND})\ \ a \neq b \rightarrow \Box a \neq b$$

If we combine the **tree rules** for K with those for classical predicate logic, we can prove (NI) but not (ND).

New rule:

$$\begin{array}{ll} \eta_1 = \eta_2 & (\omega) \\ \vdots & \\ \eta_1 = \eta_2 & (\nu) \end{array}$$

↑ old

The necessity of identity and distinctness

The following sentences are CK-valid and VK-valid:

(**NI**) $a = b \rightarrow \Box a = b$

(**ND**) $a \neq b \rightarrow \Box a \neq b$

If we combine the **axiomatic rules** for K with those for classical predicate logic, we can prove (NI) but not (ND).

New axiom:

(**ND**) $\forall x \forall y (x \neq y \rightarrow \Box x \neq y)$

The necessity of identity and distinctness

There appear to be counterexamples to (NI):

Hesperus = Phosphorus.

Hammurabi knows that Hesperus = Phosphorus.

The necessity of identity and distinctness

There appear to be counterexamples to (ND):

$$\frac{\text{William Shakespeare} \neq \text{Francis Bacon.}}{\text{Lucy knows that William Shakespeare} \neq \text{Francis Bacon.}}$$

The necessity of identity and distinctness

There appear to be counterexamples to (ND):

Raiātea and Tahaa are different islands, but they might have been a single island.

$Ir \wedge It \wedge r \neq t \wedge \Diamond \exists x (Ix \wedge x = r \wedge x = t)$

This entails:

$r \neq t \wedge \Diamond (r = t)$

Thinking about individuals

Thinking about individuals

Leibniz' Law:

$$\frac{\begin{array}{l} A \\ b = c \end{array}}{A[c//b]}$$

Thinking about individuals

Apparent counterexample:

Hammurabi knows that Hesperus is visible in the evening sky.
Hesperus = Phosphorus.

Hammurabi knows that Phosphorus is visible in the evening sky.

$\Box Vh$
$h = p$

$\Box Vp$

Thinking about individuals

We have assumed a **referential semantics** in which the meaning of a name is simply an individual.

$V(a) = \text{Alice.}$
$V(b) = \text{Bob.}$

This renders Leibniz' Law valid.

$$\frac{\begin{array}{l} A \\ b = c \end{array}}{A[c//b]}$$

But it seems to make false predictions about epistemic modality.

The bullet-biting response:

Hammurabi really did know that Phosphorus is visible in the evening sky.

Follow-up problem:

- Hammurabi believed that Phosphorus is not visible in the evening sky.
- On the bullet-biting account, Hammurabi had inconsistent beliefs.
- We can’t use Kripke semantics to model inconsistent beliefs.

Thinking about individuals

Russell's (1905) response:

"...proper names are usually really descriptions. That is to say, the thought in the mind of a person using a proper name correctly can only be expressed explicitly if we replace the proper name by a description."

Russell's (1905) response:

Hammurabi knows that Hesperus is visible in the evening sky.
Hesperus = Phosphorus.

Hammurabi knows that Phosphorus is visible in the evening sky.

$\Box\exists x(Hx \wedge \forall y(Hy \rightarrow x{=}y) \wedge Vx)$
$h = p$ or $\exists x(Hx \wedge \forall y(Hy \rightarrow x{=}y) \wedge \exists y(Py \wedge \forall z(Pz \rightarrow z{=}y) \wedge x{=}y))$

$\Box\exists x(Px \wedge \forall y(Py \rightarrow x{=}y) \wedge Vx)$

Thinking about individuals

Frege's response:

An individual can play many roles.

Every name is associated (not just with an individual but) with a role.

At every world, the name picks out whatever plays the associated role.

The role associated with 'Hesperus' is being the brightest body in the evening sky.

The role associated with 'Hesperus' is being the brightest body in the evening sky.

At our world, Venus plays both of these roles.

At other worlds, different things play the two roles.

Frege's response:

Leibniz' Law is invalid.

$\Box Vh$

$h = p$

$\Box Vp$

Individual Concept Semantics

Individual Concept Semantics

We now assume that the meaning a name is a role.

- being the brightest body in the morning sky
- being the brightest body in the evening sky
- being the inventor of the zip
- ...

Such a role can be represented by a function from worlds to individuals.

Functions from worlds to indivuals are called **individual concepts** or **intensional objects**.

Individual Concept Semantics

In **individual concept semantics**, the interpretation function assigns to every name an individual concept.

$$V(a, w) = Venus$$
$$V(a, v) = Jupiter$$

$M, w \models Fa$ iff $V(a, w) \in V(F, w)$.

Leibniz' Law is no longer valid for modal sentences.

$$\frac{\begin{array}{l}\Box Vh \\ h = p\end{array}}{\Box Vp}$$

Individual Concept Semantics

In **individual concept semantics**, the interpretation function assigns to every name an individual concept.

$$V(a, w) = Venus$$
$$V(a, v) = Jupiter$$

$M, w \models Fa$ iff $V(a, w) \in V(F, w)$.

The "necessity of identity" and the "necessity of distinctness" are also invalid.

(NI) $a = b \rightarrow \Box(a = b)$

(ND) $a \neq b \rightarrow \Box(a \neq b)$.

Individual Concept Semantics

Problems:

1. There is no complete proof procedure for this semantics.
2. $\Box\exists x A \rightarrow \exists x \Box A$ becomes valid.
3. Is every name associated with a unique role?

"Mary Ann Evans is George Elliot, but Smith doesn't know that she is."

More puzzles

More puzzles

There are two tickets, numbered 1 and 2; one is blue, one is red; we don't know which colour goes with which number. We know that the blue ticket won.

1. Ticket 1 might be the winner. ($\Diamond Wt_1$)
2. Ticket 2 might be the winner. ($\Diamond Wt_2$)
3. These are all the tickets. ($\forall x(Tx \to (x{=}t_1 \lor x{=}t_2)))$
4. So: Any ticket might be the winner. ($\forall x(Tx \to \Diamond Wx)$)
5. The red ticket is a ticket. (Tr)
6. So: The red ticket might be the winner. ($\Diamond Wr$)

More puzzles

'If I were you I wouldn't accept the offer.'

$a = b \mathrel{\Box\!\!\rightarrow} Aa$

Here we need worlds in which I = you. What do these worlds look like?

More puzzles

Alice the time travel is about to travel back in time to meet her younger self.

As she arrives, is she young or old?

$\mathrm{P}\, Fa$?
$\mathrm{P}\, \neg Fa$?
$\mathrm{P}(Fa \wedge \neg Fa)$?