Logic 2: Modal Logic

Lecture 19

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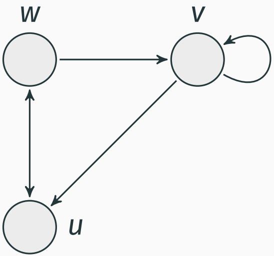
Models for modal predicate logic

Models for modal predicate logic



Where is Fa true?

Where is $Fa \wedge Fb$ true?

Where is $\Diamond Fa$ true?

Where is $\forall x Fx$ true?

Where is $\Diamond \forall x Fx$ true?

Where is $\forall x \Diamond Fx$ true?

$V(a) = Alice$

$V(b) = Bob$

$V(F, w) = \{Alice, Bob\}$

$V(F, v) = \{Alice\}$

$V(F, u) = \{Bob\}$

$D_w = \{Alice, Bob\}$

$D_v = \{Alice, Bob\}$

$D_u = \{Alice, Bob\}$

Models for modal predicate logic

A model contains just enough information to tell us which sentences of $\mathfrak{L}_P$ are true at any world.

A model consists of

- a set W of worlds,
- an accessibility relation R on W,
- for each world w a domain D_w of individuals,
- an interpretation function V that
  - assigns to every name an individual and
  - to every predicate and world a set of (tuples of) individuals.

Models for modal predicate logic

A sentence is **valid** iff it is true at all worlds in all models.

We can define different concepts of validity (different logics) by imposing constraints on the models.

- Every world has access to some world.
- Every world is accessible from itself.
- ...
- Every world has the same domain of individuals.

Let's explore this last option.

Modal predicate logic with constant domains

Modal predicate logic with constant domains

A sentence is **CK-valid** iff it is true at all worlds in all models in which the domain of individuals is constant across worlds.

If we combine the tree rules for classical predicate logic with those of K we get a sound and almost complete proof method for CK-validity.

Modal predicate logic with constant domains

Target: $\Box(Fa \rightarrow Fb) \rightarrow (\Box Fa \rightarrow \Box Fb)$

1. $\neg(\Box(Fa \rightarrow Fb) \rightarrow (\Box Fa \rightarrow \Box Fb))$ (w) (Ass.)

Modal predicate logic with constant domains

Target: $\Box(Fa \rightarrow Fb) \rightarrow (\Box Fa \rightarrow \Box Fb)$

1.	$\neg(\Box(Fa \rightarrow Fb) \rightarrow (\Box Fa \rightarrow \Box Fb))$	(w)	(Ass.)
2.	$\Box(Fa \rightarrow Fb)$	(w)	(1)
3.	$\neg(\Box Fa \rightarrow \Box Fb)$	(w)	(1)

Modal predicate logic with constant domains

Target: $\Box(Fa \rightarrow Fb) \rightarrow (\Box Fa \rightarrow \Box Fb)$

1.	$\neg(\Box(Fa \rightarrow Fb) \rightarrow (\Box Fa \rightarrow \Box Fb))$	(w)	(Ass.)
2.	$\Box(Fa \rightarrow Fb)$	(w)	(1)
3.	$\neg(\Box Fa \rightarrow \Box Fb)$	(w)	(1)
4.	$\Box Fa$	(w)	(3)
5.	$\neg\Box Fb$	(w)	(3)

Modal predicate logic with constant domains

Target: $\Box(Fa \rightarrow Fb) \rightarrow (\Box Fa \rightarrow \Box Fb)$

1.	$\neg(\Box(Fa \rightarrow Fb) \rightarrow (\Box Fa \rightarrow \Box Fb))$	(w)	(Ass.)
2.	$\Box(Fa \rightarrow Fb)$	(w)	(1)
3.	$\neg(\Box Fa \rightarrow \Box Fb)$	(w)	(1)
4.	$\Box Fa$	(w)	(3)
5.	$\neg\Box Fb$	(w)	(3)
6.	wRv		(5)
7.	$\neg Fb$	(v)	(5)

Modal predicate logic with constant domains

Target: $\Box(Fa \to Fb) \to (\Box Fa \to \Box Fb)$

1.	$\neg(\Box(Fa \to Fb) \to (\Box Fa \to \Box Fb))$	(w)	(Ass.)
2.	$\Box(Fa \to Fb)$	(w)	(1)
3.	$\neg(\Box Fa \to \Box Fb)$	(w)	(1)
4.	$\Box Fa$	(w)	(3)
5.	$\neg\Box Fb$	(w)	(3)
6.	wRv		(5)
7.	$\neg Fb$	(v)	(5)
8.	$Fa \to Fb$	(v)	(2,6)

Modal predicate logic with constant domains

Target: $\Box(Fa \to Fb) \to (\Box Fa \to \Box Fb)$

1.	$\neg(\Box(Fa \to Fb) \to (\Box Fa \to \Box Fb))$	(w)	(Ass.)
2.	$\Box(Fa \to Fb)$	(w)	(1)
3.	$\neg(\Box Fa \to \Box Fb)$	(w)	(1)
4.	$\Box Fa$	(w)	(3)
5.	$\neg\Box Fb$	(w)	(3)
6.	wRv		(5)
7.	$\neg Fb$	(v)	(5)
8.	$Fa \to Fb$	(v)	(2,6)
9.	Fa	(v)	(4,6)

Modal predicate logic with constant domains

Target: $\Box(Fa \to Fb) \to (\Box Fa \to \Box Fb)$

1.	$\neg(\Box(Fa \to Fb) \to (\Box Fa \to \Box Fb))$	(w)	(Ass.)
2.	$\Box(Fa \to Fb)$	(w)	(1)
3.	$\neg(\Box Fa \to \Box Fb)$	(w)	(1)
4.	$\Box Fa$	(w)	(3)
5.	$\neg\Box Fb$	(w)	(3)
6.	wRv		(5)
7.	$\neg Fb$	(v)	(5)
8.	$Fa \to Fb$	(v)	(2,6)
9.	Fa	(v)	(4,6)

10. $\neg Fa$ (v) (8) | 11. Fb (v) (8)

Modal predicate logic with constant domains

Target: $\forall x \Box Fx \to \Box \forall x Fx$

1.	$\neg(\forall x \Box Fx \to \Box \forall x Fx)$	(w)	(Ass.)

Modal predicate logic with constant domains

Target: $\forall x \Box Fx \to \Box \forall x Fx$

1.	$\neg(\forall x \Box Fx \to \Box \forall x Fx)$	(w)	(Ass.)
2.	$\forall x \Box Fx$	(w)	(1)
3.	$\neg \Box \forall x Fx$	(w)	(1)

Modal predicate logic with constant domains

Target: $\forall x \Box Fx \to \Box \forall x Fx$

1.	$\neg(\forall x \Box Fx \to \Box \forall x Fx)$	(w)	(Ass.)
2.	$\forall x \Box Fx$	(w)	(1)
3.	$\neg \Box \forall x Fx$	(w)	(1)
4.	wRv		(3)
5.	$\neg \forall x Fx$	(v)	(3)

Modal predicate logic with constant domains

Target: $\forall x \Box Fx \rightarrow \Box \forall x Fx$

1.	$\neg(\forall x \Box Fx \rightarrow \Box \forall x Fx)$	(w)	(Ass.)
2.	$\forall x \Box Fx$	(w)	(1)
3.	$\neg \Box \forall x Fx$	(w)	(1)
4.	wRv		(3)
5.	$\neg \forall x Fx$	(v)	(3)
6.	$\neg Fa$	(v)	(5)

Modal predicate logic with constant domains

Target: $\forall x \Box Fx \rightarrow \Box \forall x Fx$

1.	$\neg(\forall x \Box Fx \rightarrow \Box \forall x Fx)$	(w)	(Ass.)
2.	$\forall x \Box Fx$	(w)	(1)
3.	$\neg \Box \forall x Fx$	(w)	(1)
4.	wRv		(3)
5.	$\neg \forall x Fx$	(v)	(3)
6.	$\neg Fa$	(v)	(5)
7.	$\Box Fa$	(w)	(2)

Modal predicate logic with constant domains

Target: $\forall x \Box Fx \rightarrow \Box \forall x Fx$

1.	$\neg(\forall x \Box Fx \rightarrow \Box \forall x Fx)$	(w)	(Ass.)
2.	$\forall x \Box Fx$	(w)	(1)
3.	$\neg \Box \forall x Fx$	(w)	(1)
4.	wRv		(3)
5.	$\neg \forall x Fx$	(v)	(3)
6.	$\neg Fa$	(v)	(5)
7.	$\Box Fa$	(w)	(2)
8.	Fa	(v)	(7,4)
	x		

Modal predicate logic with constant domains

(**BF**) $\forall x \Box A \to \Box \forall x A$

(**CBF**) $\Box \forall x A \to \forall x \Box A$

The Barcan Formula (BF) and the Converse Barcan Formula (CBF) are CK-valid.

(BF) corresponds to the assumption that if wRv then every member of D_v is a member of D_w.

1. Suppose unicorns could have existed, but nothing that actually exists could have been a unicorn.
2. Then $\forall x \Box \neg Ux$ is true.
3. But $\Box \forall x \neg Ux$ is false.

Modal predicate logic with constant domains

(**BF**) $\forall x \Box A \rightarrow \Box \forall x A$

(**CBF**) $\Box \forall x A \rightarrow \forall x \Box A$

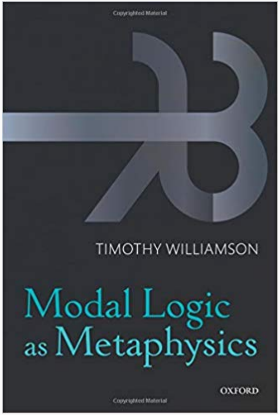
The Barcan Formula (BF) and the Converse Barcan Formula (CBF) are CK-valid.

medskip

(CBF) corresponds to the assumption that if wRv then every member of D_w is a member of D_v.

1. Suppose you could have failed to exist. Let E be a property that applies to d at w iff $d \in D_w$.
2. Then $\Box \forall x E x$ is true.
3. But $\forall x \Box E x$ is false.

Modal predicate logic with constant domains



Necessitism:

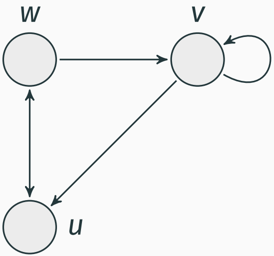
- Everything necessarily exists.
- Nothing could have failed to exist.
- If your parents had never met, you would still have existed, but you would not have been a person.

Permanentism:

- Everything has always existed and will always exist.
- Anything that ever existed or will exist exists now.
- The dinosaurs still exist, but they are no longer dinosaurs.

Modal predicate logic with variable domains

Modal predicate logic with variable domains



Is $\Diamond Fa$ true at w?

Is $\Box Fa$ true at w?

Is Fa true at u?

Is $\forall x Fx$ true at u?

$V(a) = Alice$

$V(b) = Bob$

$V(F, w) = \{Alice, Bob\}$

$V(F, v) = \{Alice\}$

$V(F, u) = \{Bob\}$

$D_w = \{Alice, Bob\}$

$D_v = \{Alice, Bob, Carol\}$

$D_u = \{Bob\}$

Modal predicate logic with variable domains

The rule of Universal Instantiation appears to be invalid in variable-domain models.

$\forall x A$

$\vdots$

$A[c/x]$

↑

old or first

Revised rule:

$\forall x A \quad (\omega)$

$\vdots$

$c \neq c \quad (\omega) \qquad A[c/x] \quad (\omega)$

↑

old

Modal predicate logic with variable domains

If we have variable domains, we effectively allow for empty names.

Logics with empty names are called **free logics**.

In free logic, $\forall x Fx$ does not entail Fa.

Some free logics are three-valued: neither Fb nor $\neg Fb$ may be true.