Logic 2: Modal Logic

Lecture 14

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Review

Review

Temporal operators:

- F: It will (at some point) be the case that
- G: It is always going to be the case that
- P: It was (at some point) the case that
- H: It has always been the case that

Review

Temporal Model

A **temporal model** consists of

- a non-empty set T (of “times”),
- a binary relation $<$ on T (the **precedence relation**),
- a function V that assigns to each sentence letter a subset of T.

Review

Standard Temporal Semantics

(g) $M, t \models \mathrm{F}A$ iff $M, s \models A$ for some s such that $t < s$.

(h) $M, t \models \mathrm{G}A$ iff $M, s \models A$ for all s such that $t < s$.

(i) $M, t \models \mathrm{P}A$ iff $M, s \models A$ for some s such that $s < t$.

(j) $M, t \models \mathrm{H}A$ iff $M, s \models A$ for all s such that $s < t$.

Review

The logic of time depends on formal properties of the precedence relation.

- We always assume that $>$ is transitive. This renders $\mathrm{G}A \to \mathrm{G}\mathrm{G}A$ valid.
- We might assume that $>$ is asymmetric and irreflexive. This doesn't affect the logic.
- We might assume that $>$ is discrete. This would render $(A \wedge \mathrm{H}A) \to \mathrm{F}\mathrm{H}A$ valid.
- We might assume that $>$ is connected. This would render $\mathrm{F}\mathrm{P}A \to (\mathrm{F}A \vee A \vee \mathrm{P}A)$ and $\mathrm{P}\mathrm{F}A \to (\mathrm{P}A \vee A \vee \mathrm{F}A)$ valid.

Review

"Asymmetry, irreflexivity and connectedness correspond to nothing".

A property X of the accessibility relation corresponds to a schema A iff

- on every X frame, every instance of A is valid, and
- on every non-X frame, some instance of A is invalid.

Universality corresponds to nothing.

But requiring universality changes the logic!

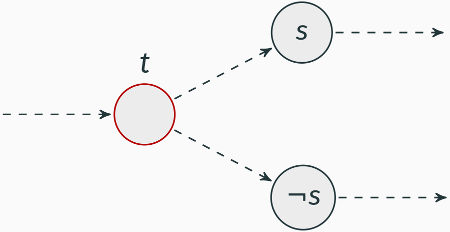
- Every instance of $\Box A \to A$ is valid on every universal frame.

Asymmetry usually doesn't change the logic.

- If A is valid on every asymmetric frame then A is valid on every frame.

Branching time

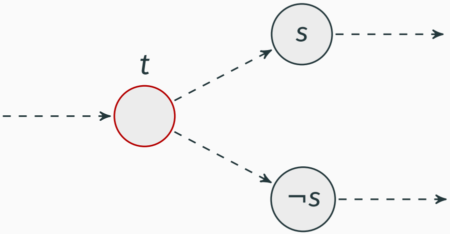
Branching time



Is Fs true at t? Yes.

Intuition: ‘There will be a sea battle’ is not true at t.

Branching time



Standard semantics:

$M, t \models \mathsf{F} A$ iff $M, t' \models A$ for some t' such that $t < t'$
iff A is true at some future point on some history through t.

“Peircean” semantics (CTL):

$M, t \models \mathsf{F} A$ iff A is true at some future point on every history through t.

Is $\mathsf{F}\, s$ true at t in Peircean semantics? No.

Branching time

Standard semantics:

$M, t \models \mathsf{F}A$ iff A is true at some future point on some history through t.

“Peircean” semantics (CTL):

$M, t \models \mathsf{F}A$ iff A is true at some future point on every history through t.

We can factor out the quantification over histories.

$\Box A$: On every history (through the present point) ...
$\Diamond A$: On some history (through the present point) ...

$\Diamond \mathsf{F}A$: A is true at some future point on some history through t.
$\Box \mathsf{F}A$: A is true at some future point on every history through t.

Branching time

$\Box A$: On every history (through the present point) ...
$\Diamond A$: On some history (through the present point) ...
F A: At some point in the future ...

How does this language work?

$M, t \models \Diamond A$ iff ?
$M, t \models \mathrm{F} A$ iff ?

"Ockhamist" semantics (CTL*):

$M, h, t \models \Diamond A$ iff $M, h', t \models A$ for some history h' that contains t.
$M, h, t \models \mathrm{F} A$ iff $M, h, t' \models A$ for some t' with $t < t'$.

Branching time

"Ockhamist" semantics (CTL*)

$$M, h, t \models \Diamond A \quad \text{iff } M, h', t \models A \text{ for some history } h' \text{ that contains } t.$$
$$M, h, t \models \mathsf{F} A \quad \text{iff } M, h, t' \models A \text{ for some } t' \text{ with } t < t'.$$

Truth is defined relative to three **parameters**: M, h, t.

Only M and t represent a scenario and an interpretation.

Ockhamism doesn't tell us which sentences are true in a given scenario under a given interpretation.

So it doesn't tell us which sentences are true in all scenarios under all interpretations.

Branching time

"Ockhamist" semantics (CTL*)

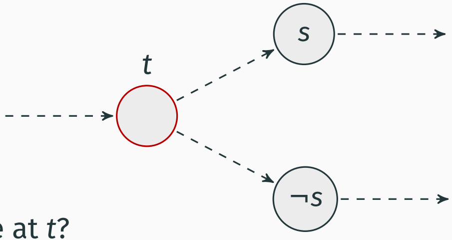
$M, h, t \models \Diamond A$ iff $M, h', t \models A$ for some history h' that contains t.
$M, h, t \models \mathsf{F}A$ iff $M, h, t' \models A$ for some t' with $t < t'$.

Supervaluationism:

$M, t \models A$ iff $M, h, t \models A$ for every history h through t.

Branching time

$M, t \models A$ iff $M, h, t \models A$ for every history h through t.



Which of these are true at t?

- $\Diamond \mathsf{F} s$
- $\Box \mathsf{F} s$
- $\mathsf{F} s$
- $\neg \mathsf{F} s$
- $\mathsf{F} s \vee \neg \mathsf{F} s$

Branching time

Supervaluationist Ockhamism determines a **three-valued logic**. A sentence can be

- true
- false
- neither

The truth-value of a truth-functionally complex sentence at a scenario is not determined by the truth-value of the parts:

- Fs and $\neg Fs$ are neither true nor false, $Fs \vee \neg Fs$ is true.
- Fs and Fs are neither true nor false, $Fs \vee Fs$ is neither true nor false.

Branching time

In other three-valued logics, the truth-value of truth-functionally complex sentences is determined by the truth-values of the parts:

A	B	$A \vee B$
1	1	1
1	N	1
1	0	1
N	1	1
N	N	N
N	0	N
0	1	1
0	N	N
0	0	0