# Logic 2: Modal Logic

## Lecture 13

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# Temporal logic

# Temporal logic


$F\,r$

$F\,\neg r$?

$F\,F\,s$?

$F\,F\,r$?

$P\,F\,s$?

F: at some point in the future

P: at some point in the past

# Temporal logic

**Temporal operators:**

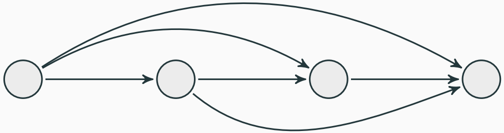
- F: It will (at some point) be the case that
- P: It was (at some point) the case that

- G: It is always going to be the case that
- H: It has always been the case that

# Temporal logic

The dashed arrows represent the “next point in time” relation (assuming that time is discrete).

Let’s define a new relation that holds between $t$ and $s$ iff $t$ is earlier than $s$.



This relation is called the **precedence relation** and usually expressed by ’<’.

**Temporal Model**

A **temporal model** consists of

- a non-empty set $T$ (of “times”),
- a binary relation $<$ on $T$ (the precedence relation),
- a function $V$ that assigns to each sentence letter of $\mathfrak{L}_T$ a subset of $T$.

Temporal models are Kripke models.

# Temporal logic

## Standard Temporal Semantics

| | | |
|---|---|---|
| (a) | $M, t \models \rho$ | iff $t$ is in $V(\rho)$. |
| (b) | $M, t \models \neg A$ | iff $M, t \not\models A$. |
| (c) | $M, t \models A \wedge B$ | iff $M, t \models A$ and $M, t \models B$. |
| (d) | $M, t \models A \vee B$ | iff $M, t \models A$ or $M, t \models B$. |
| (e) | $M, t \models A \rightarrow B$ | iff $M, t \models B$ or $M, t \not\models A$. |
| (f) | $M, t \models A \longleftrightarrow B$ | iff $M, t \models (A \rightarrow B)$ and $M, t \models (B \rightarrow A)$. |
| (g) | $M, t \models \mathrm{F}A$ | iff $M, s \models A$ for some $s$ in $T$ such that $t < s$. |
| (h) | $M, t \models \mathrm{G}A$ | iff $M, s \models A$ for all $s$ in $T$ such that $t < s$. |
| (i) | $M, t \models \mathrm{P}A$ | iff $M, s \models A$ for some $s$ in $T$ such that $s < t$. |
| (j) | $M, t \models \mathrm{H}A$ | iff $M, s \models A$ for all $s$ in $T$ such that $s < t$. |

# Temporal logic

We may want to put constraints on the precedence relation.

(T) $\mathrm{G}A \rightarrow A$ (Reflexivity)

(D) $\mathrm{G}A \rightarrow \mathrm{F}A$ (Seriality)

(B) $A \rightarrow \mathrm{G}\mathrm{F}A$ (Symmetry)

(4) $\mathrm{G}A \rightarrow \mathrm{G}\mathrm{G}A$ (Transitivity)

(5) $\mathrm{F}A \rightarrow \mathrm{G}\mathrm{F}A$ (Euclidity)

(G) $\mathrm{F}\mathrm{G}A \rightarrow \mathrm{G}\mathrm{F}A$ (Convergence)

# The flow of time

# The flow of time

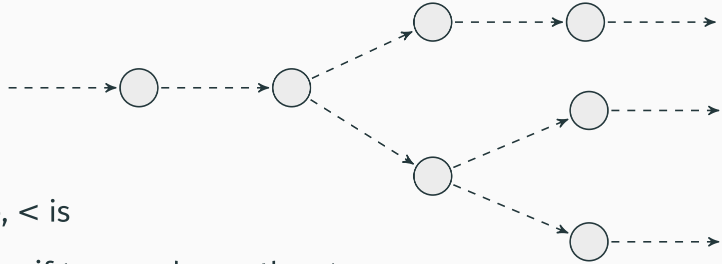
In this frame, $<$ is

- transitive: if $t < s$ and $s < r$ then $t < r$.
- asymmetric: if $t < s$ then $s \not< t$.
- irreflexive: for no time, $t < t$.
- discrete: if $t < s$ then there is an $r$ such that $t < r$ and for no $x$, $t < x < r$.
- connected: for every $t, s$, either $t < s$ or $t = s$ or $s < t$.

# The flow of time

- Transitivity corresponds to (4): $\mathrm{G}A \to \mathrm{GG}A$.
- Asymmetry corresponds to nothing.
- Irreflexivity corresponds to nothing.
- Discreteness corresponds to $(A \wedge HA) \to FHA$.
- Connectedness corresponds to nothing but it renders $(\mathrm{FP}A \to (\mathrm{F}A \vee A \vee \mathrm{P}A)) \wedge (\mathrm{PF}A \to (\mathrm{P}A \vee A \vee \mathrm{F}A))$ valid.

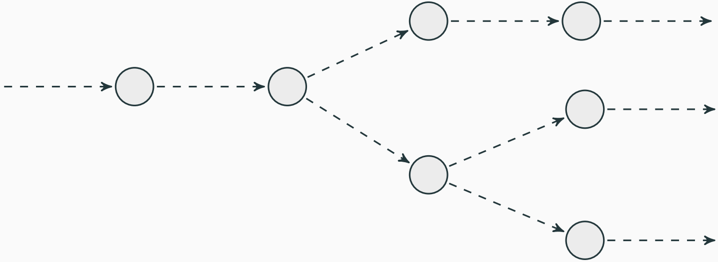
# The flow of time



In this frame, $<$ is

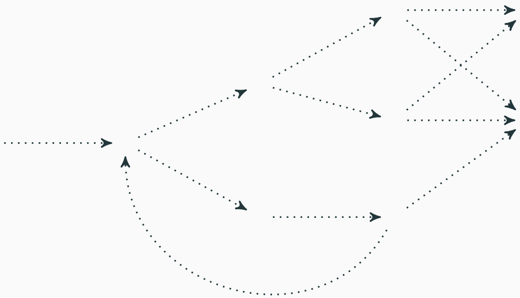
- transitive: if $t < s$ and $s < r$ then $t < r$.
- asymmetric: if $t < s$ then $s \not< t$.
- discrete: if $t < s$ then there is an $r$ such that $t < r$ and for no $x$, $t < x < r$.
- left-linear: if $t < r$ and $s < r$ then either $t < s$ or $t = s$ or $s < t$.
- right-branching: for some $r < t$ and $r < s$, neither $t < s$ or $t = s$ or $s < t$.

# The flow of time



- Left-linearity corresponds to $\mathrm{F}\,\mathrm{P}A \to (\mathrm{F}A \lor A \lor \mathrm{P}A)$.
- Right-branchingness corresponds to nothing.

Relativistic time:



# Trees

# Trees

$$\begin{array}{ll} \mathsf{G}A & (\omega) \\ \omega < \nu & \\ \vdots & \\ A & (\nu) \\ & \uparrow \\ & \text{old} \end{array} \qquad \begin{array}{ll} \mathsf{F}A & (\omega) \\ \vdots & \\ \omega < \nu & \\ A & (\nu) \\ & \uparrow \\ & \text{new} \end{array} \qquad \begin{array}{ll} \mathsf{H}A & (\omega) \\ \color{red}{\nu < \omega} & \\ \vdots & \\ A & (\nu) \\ & \uparrow \\ & \text{old} \end{array} \qquad \begin{array}{ll} \mathsf{P}A & (\omega) \\ \vdots & \\ \color{red}{\nu < \omega} & \\ A & (\nu) \\ & \uparrow \\ & \text{new} \end{array}$$

$$\begin{array}{ll} \neg\,\mathsf{G}A & (\omega) \\ \vdots & \\ \omega < \nu & \\ \neg A & (\nu) \\ & \uparrow \\ & \text{new} \end{array} \qquad \begin{array}{ll} \neg\,\mathsf{F}A & (\omega) \\ \omega < \nu & \\ \vdots & \\ \neg A & (\nu) \\ & \uparrow \\ & \text{old} \end{array} \qquad \begin{array}{ll} \neg\,\mathsf{H}A & (\omega) \\ \vdots & \\ \color{red}{\nu < \omega} & \\ \neg A & (\nu) \\ & \uparrow \\ & \text{new} \end{array} \qquad \begin{array}{ll} \neg\,\mathsf{P}A & (\omega) \\ \color{red}{\nu < \omega} & \\ \vdots & \\ \neg A & (\nu) \\ & \uparrow \\ & \text{old} \end{array}$$

# Trees

Left-linearity

$\nu < \omega$

$\upsilon < \omega$

$\nu < \upsilon \qquad \nu = \upsilon \qquad \upsilon < \nu$

Right-linearity

$\omega < \nu$

$\omega < \upsilon$

$\nu < \upsilon \qquad \nu = \upsilon \qquad \upsilon < \nu$

$A \quad (\omega)$

$\omega = \nu$

$A \quad (\nu)$

$A \quad (\omega)$

$\nu = \omega$

$A \quad (\nu)$

# A puzzle

# A puzzle

## Operators

| | |
|---|---|
| $\mathrm{P}\,A$ | it was once the case that $A$ |
| $\mathrm{F}\,A$ | it will once be the case that $A$ |
| $\Diamond A$ | we can bring it about that $A$ |

## Assumptions

(i) $\mathrm{F}\,A \models \mathrm{P}\,\mathrm{F}\,A$

(ii) $\neg\,\mathrm{P}\,A \models \neg\Diamond\,\mathrm{P}\,A$

(iii) If $A \models B$ then $\Diamond A \models \Diamond B$

1. Suppose $q$ neither is, nor was, nor will ever be the case.
2. So $\neg\,\mathrm{P}\,\mathrm{F}\,q$.
3. Then $\neg\Diamond\,\mathrm{P}\,\mathrm{F}\,q$, by (ii).
4. But $Fq$ entails $\mathrm{P}\,\mathrm{F}\,q$, by (i).
5. So $\Diamond Fq$ entails $\Diamond\,\mathrm{P}\,\mathrm{F}\,q$, by (iii).
6. So $\neg\Diamond\,\mathrm{F}\,q$, by (3), (5), and Modus Tollens.