Logic 2: Modal Logic

Lecture 13

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Temporal logic

Temporal logic

$\mathrm{F}\, r$

$\mathrm{F} \neg r$?

$\mathrm{F}\,\mathrm{F}\, s$?

$\mathrm{F}\,\mathrm{F}\, r$?

$\mathrm{P}\,\mathrm{F}\, s$?

F: at some point in the future

P: at some point in the past

Temporal logic

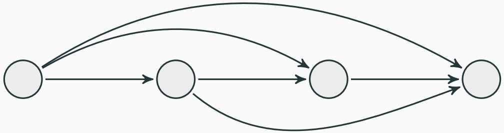
Temporal operators:

- F: It will (at some point) be the case that
- P: It was (at some point) the case that
- G: It is always going to be the case that
- H: It has always been the case that

Temporal logic

The dashed arrows represent the “next point in time” relation (assuming that time is discrete).

Let’s define a new relation that holds between *t* and *s* iff *t* is earlier than *s*.



This relation is called the **precedence relation** and usually expressed by ’<’.

Temporal Model

A **temporal model** consists of

- a non-empty set T (of "times"),
- a binary relation $<$ on T (the precedence relation),
- a function V that assigns to each sentence letter of $\mathfrak{L}_T$ a subset of T.

Temporal models are Kripke models.

Standard Temporal Semantics

(a)	$M, t \models \rho$	iff t is in $V(\rho)$.
(b)	$M, t \models \neg A$	iff $M, t \not\models A$.
(c)	$M, t \models A \wedge B$	iff $M, t \models A$ and $M, t \models B$.
(d)	$M, t \models A \vee B$	iff $M, t \models A$ or $M, t \models B$.
(e)	$M, t \models A \rightarrow B$	iff $M, t \models B$ or $M, t \not\models A$.
(f)	$M, t \models A \longleftrightarrow B$	iff $M, t \models (A \rightarrow B)$ and $M, t \models (B \rightarrow A)$.
(g)	$M, t \models \mathrm{F}A$	iff $M, s \models A$ for some s in T such that $t < s$.
(h)	$M, t \models \mathrm{G}A$	iff $M, s \models A$ for all s in T such that $t < s$.
(i)	$M, t \models \mathrm{P}A$	iff $M, s \models A$ for some s in T such that $s < t$.
(j)	$M, t \models \mathrm{H}A$	iff $M, s \models A$ for all s in T such that $s < t$.

Temporal logic

We may want to put constraints on the precedence relation.

(T) $\mathrm{G}A \rightarrow A$ (Reflexivity)

(D) $\mathrm{G}A \rightarrow \mathrm{F}A$ (Seriality)

(B) $A \rightarrow \mathrm{G}\mathrm{F}A$ (Symmetry)

(4) $\mathrm{G}A \rightarrow \mathrm{G}\mathrm{G}A$ (Transitivity)

(5) $\mathrm{F}A \rightarrow \mathrm{G}\mathrm{F}A$ (Euclidity)

(G) $\mathrm{F}\mathrm{G}A \rightarrow \mathrm{G}\mathrm{F}A$ (Convergence)

The flow of time

The flow of time

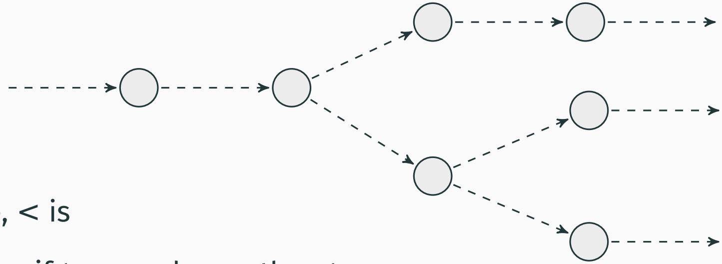
In this frame, $<$ is

- transitive: if $t < s$ and $s < r$ then $t < r$.
- asymmetric: if $t < s$ then $s \not< t$.
- irreflexive: for no time, $t < t$.
- discrete: if $t < s$ then there is an r such that $t < r$ and for no x, $t < x < r$.
- connected: for every t, s, either $t < s$ or $t = s$ or $s < t$.

The flow of time

- Transitivity corresponds to (4): $\mathsf{G}A \to \mathsf{G}\mathsf{G}A$.
- Asymmetry corresponds to nothing.
- Irreflexivity corresponds to nothing.
- Discreteness corresponds to $(A \wedge HA) \to FHA$.
- Connectedness corresponds to nothing but it renders $(\mathsf{F}\mathsf{P}A \to (\mathsf{F}A \vee A \vee \mathsf{P}A)) \wedge (\mathsf{P}\mathsf{F}A \to (\mathsf{P}A \vee A \vee \mathsf{F}A))$ valid.

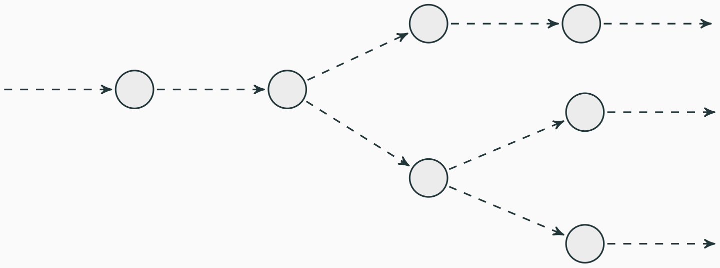
The flow of time



In this frame, $<$ is

- transitive: if $t < s$ and $s < r$ then $t < r$.
- asymmetric: if $t < s$ then $s \nless t$.
- discrete: if $t < s$ then there is an r such that $t < r$ and for no x, $t < x < r$.
- left-linear: if $t < r$ and $s < r$ then either $t < s$ or $t = s$ or $s < t$.
- right-branching: for some $r < t$ and $r < s$, neither $t < s$ or $t = s$ or $s < t$.

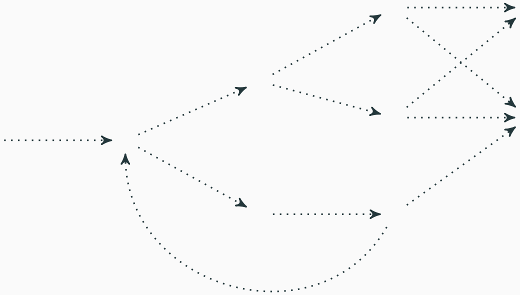
The flow of time



- Left-linearity corresponds to $\mathsf{F}\,\mathsf{P}\,A \rightarrow (\mathsf{F}\,A \vee A \vee \mathsf{P}\,A)$.
- Right-branchingness corresponds to nothing.

The flow of time

Relativistic time:



Trees

Trees

$\mathsf{G}A \quad (\omega)$
$\omega < \nu$
$\vdots$
$A \quad (\nu)$
↑ old

$\mathsf{F}A \quad (\omega)$
$\vdots$
$\omega < \nu$
$A \quad (\nu)$
↑ new

$\mathsf{H}A \quad (\omega)$
$\nu < \omega$
$\vdots$
$A \quad (\nu)$
↑ old

$\mathsf{P}A \quad (\omega)$
$\vdots$
$\nu < \omega$
$A \quad (\nu)$
↑ new

$\neg\,\mathsf{G}A \quad (\omega)$
$\vdots$
$\omega < \nu$
$\neg A \quad (\nu)$
↑ new

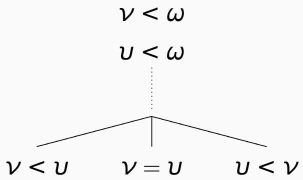
$\neg\,\mathsf{F}A \quad (\omega)$
$\omega < \nu$
$\vdots$
$\neg A \quad (\nu)$
↑ old

$\neg\,\mathsf{H}A \quad (\omega)$
$\vdots$
$\nu < \omega$
$\neg A \quad (\nu)$
↑ new

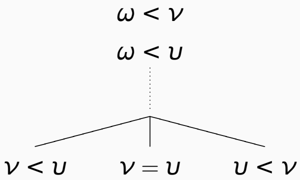
$\neg\,\mathsf{P}A \quad (\omega)$
$\nu < \omega$
$\vdots$
$\neg A \quad (\nu)$
↑ old

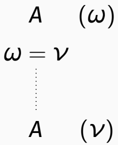
Trees

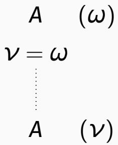
Left-linearity



Right-linearity







A puzzle

A puzzle

Operators

$\mathrm{P}\,A$	it was once the case that A
$\mathrm{F}\,A$	it will once be the case that A
$\Diamond A$	we can bring it about that A

Assumptions

(i) $\mathrm{F}\,A \models \mathrm{P}\,\mathrm{F}\,A$

(ii) $\neg\,\mathrm{P}\,A \models \neg\Diamond\,\mathrm{P}\,A$

(iii) If $A \models B$ then $\Diamond A \models \Diamond B$

1. Suppose q neither is, nor was, nor will ever be the case.
2. So $\neg\,\mathrm{P}\,\mathrm{F}\,q$.
3. Then $\neg\Diamond\,\mathrm{P}\,\mathrm{F}\,q$, by (ii).
4. But Fq entails $\mathrm{P}\,\mathrm{F}\,q$, by (i).
5. So $\Diamond Fq$ entails $\Diamond\,\mathrm{P}\,\mathrm{F}\,q$, by (iii).
6. So $\neg\Diamond\,\mathrm{F}\,q$, by (3), (5), and Modus Tollens.