Logic 2: Modal Logic

Lecture 13

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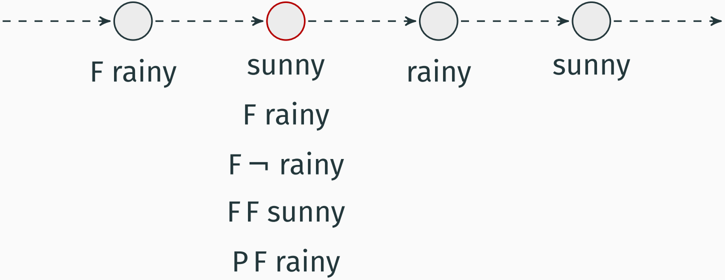
Temporal logic

Temporal logic



F: at some point in the future

P: at some point in the past

Temporal logic

Temporal operators:

- F: It will (at some point) be the case that
- P: It was (at some point) the case that
- G: It is always going to be the case that
- H: It has always been the case that

Temporal logic

Temporal Model

A **temporal model** consists of

- a non-empty set T (of "times"),
- a binary relation $<$ on T (the **precedence relation**),
- a function V that assigns to each sentence letter of $\mathfrak{L}_T$ a subset of T.

Temporal models are Kripke models.

Temporal logic

Standard Temporal Semantics

(a)	$M, t \models \rho$	iff t is in $V(\rho)$.
(b)	$M, t \models \neg A$	iff $M, t \not\models A$.
(c)	$M, t \models A \wedge B$	iff $M, t \models A$ and $M, t \models B$.
(d)	$M, t \models A \vee B$	iff $M, t \models A$ or $M, t \models B$.
(e)	$M, t \models A \rightarrow B$	iff $M, t \models B$ or $M, t \not\models A$.
(f)	$M, t \models A \longleftrightarrow B$	iff $M, t \models (A \rightarrow B)$ and $M, t \models (B \rightarrow A)$.
(g)	$M, t \models \mathrm{F}A$	iff $M, s \models A$ for some s in T such that $t < s$.
(h)	$M, t \models \mathrm{G}A$	iff $M, s \models A$ for all s in T such that $t < s$.
(i)	$M, t \models \mathrm{P}A$	iff $M, s \models A$ for some s in T such that $s < t$.
(j)	$M, t \models \mathrm{H}A$	iff $M, s \models A$ for all s in T such that $s < t$.

Temporal logic

Which of these should be valid?

(T) $\mathsf{G}A \rightarrow A$

(D) $\mathsf{G}A \rightarrow \mathsf{F}A$

(4) $\mathsf{G}A \rightarrow \mathsf{G}\mathsf{G}A$

(5) $\mathsf{F}A \rightarrow \mathsf{G}\mathsf{F}A$

(G) $\mathsf{F}\mathsf{G}A \rightarrow \mathsf{G}\mathsf{F}A$

The flow of time

The flow of time

The dashed arrows represent the "next point in time" relation.

$t < r$ iff some path along the dashed arrows leads from t to r.

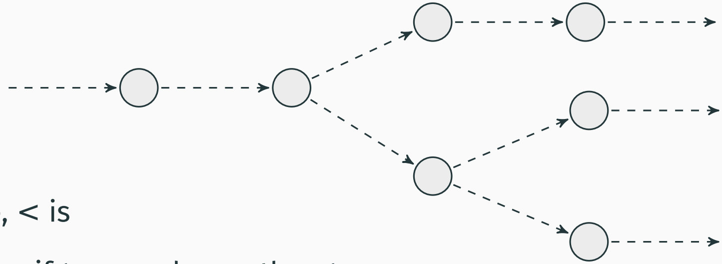
In this frame, $<$ is

- transitive: if $t < s$ and $s < r$ then $t < r$.
- asymmetric: if $t < s$ then $s \not< t$.
- irreflexive: for no time, $t < t$.
- discrete: if $t < s$ then there is an r such that $t < r$ and for no x, $t < x < r$.
- connected: for every t, s, either $t < s$ or $t = s$ or $s < t$.

The flow of time

- Transitivity corresponds to (4): $\mathsf{G}A \to \mathsf{G}\mathsf{G}A$.
- Asymmetry corresponds to nothing.
- Irreflexivity corresponds to nothing.
- Discreteness corresponds to $(A \wedge HA) \to FHA$.
- Connectedness corresponds to nothing.

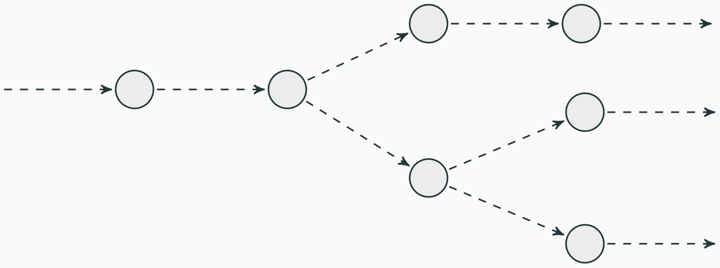
The flow of time



In this frame, $<$ is

- transitive: if $t < s$ and $s < r$ then $t < r$.
- asymmetric: if $t < s$ then $s \not< t$.
- discrete: if $t < s$ then there is an r such that $t < r$ and for no x, $t < x < r$.
- left-linear: if $t < r$ and $s < r$ then either $t < s$ or $t = s$ or $s < t$.
- right-branching: for some $r < t$ and $r < s$, neither $t < s$ or $t = s$ or $s < t$.

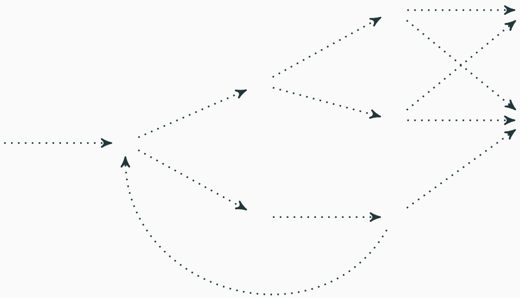
The flow of time



- Left-linearity corresponds to $\mathsf{F}\mathsf{P}A \to (\mathsf{F}A \vee A \vee \mathsf{P}A)$.
- Right-branchingness corresponds to nothing.

The flow of time

Relativistic time:



Trees

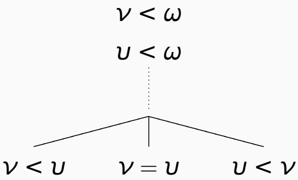
Trees

$\mathsf{G}A \quad (\omega)$	$\mathsf{F}A \quad (\omega)$	$\mathsf{H}A \quad (\omega)$	$\mathsf{P}A \quad (\omega)$
$\omega < \nu$	⋮	$\nu < \omega$	⋮
⋮	$\omega < \nu$	⋮	$\nu < \omega$
$A \quad (\nu)$	$A \quad (\nu)$	$A \quad (\nu)$	$A \quad (\nu)$
↑ old	↑ new	↑ old	↑ new

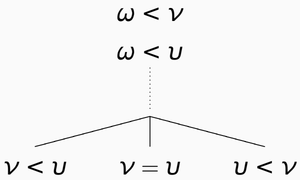
$\neg\,\mathsf{G}A \quad (\omega)$	$\neg\,\mathsf{F}A \quad (\omega)$	$\neg\,\mathsf{H}A \quad (\omega)$	$\neg\,\mathsf{P}A \quad (\omega)$
⋮	$\omega < \nu$	⋮	$\nu < \omega$
$\omega < \nu$	⋮	$\nu < \omega$	⋮
$\neg A \quad (\nu)$	$\neg A \quad (\nu)$	$\neg A \quad (\nu)$	$\neg A \quad (\nu)$
↑ new	↑ old	↑ new	↑ old

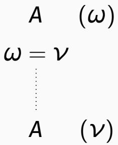
Trees

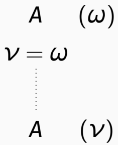
Left-linearity



Right-linearity







A puzzle

A puzzle

Operators

$\mathrm{P}\,A$	it was once the case that A
$\mathrm{F}\,A$	it will once be the case that A
$\Diamond A$	we can bring it about that A

Assumptions

(i) $\mathrm{F}\,A \models \mathrm{P}\,\mathrm{F}\,A$

(ii) $\neg\,\mathrm{P}\,A \models \neg\Diamond\,\mathrm{P}\,A$

(iii) If $A \models B$ then $\Diamond A \models \Diamond B$

1. Suppose q neither is, nor was, nor will ever be the case.
2. So $\neg\,\mathrm{P}\,\mathrm{F}\,q$.
3. Then $\neg\Diamond\,\mathrm{P}\,\mathrm{F}\,q$, by (ii).
4. But Fq entails $\mathrm{P}\,\mathrm{F}\,q$, by (i).
5. So $\Diamond Fq$ entails $\Diamond\,\mathrm{P}\,\mathrm{F}\,q$, by (iii).
6. So $\neg\Diamond\,\mathrm{F}\,q$, by (3), (5), and Modus Tollens.