Logic 2: Modal Logic

Lecture 10

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The Logic of Knowledge

The Logic of Knowledge

Possible-worlds analysis of knowledge

S knows that P iff P is true at all worlds compatible with S's knowledge.

Possible-worlds analysis of knowledge

$\mathrm{K}A$ is true at w iff A is true at all worlds accessible from w.

The logic of knowledge depends on the properties of the accessibility relation.

The Logic of Knowledge

Is the accessibility relation for knowledge reflexive?

Equivalently, is the (T)-schema valid in the logic of knowledge?

$$(\mathbf{T})\quad \mathsf{K}A \rightarrow A$$

Plausibly, yes.

We then automatically get

$$(\mathbf{D})\quad \mathsf{K}A \rightarrow \mathsf{M}A$$

The Logic of Knowledge

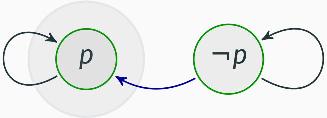
Should R be symmetric? Do we want (B) to come out valid?

$$(\mathbf{B})\quad A \rightarrow \mathrm{KM}\,A$$

Suppose you falsely believe $\neg p$.

- p is true.
- You believe that you know $\neg p$.
- You don't believe that you don't know $\neg p$.
- You don't know that you don't know $\neg p$.
- $\mathrm{K}\neg\mathrm{K}\neg p$ is false.
- $\mathrm{KM}\,p$ is false.

Also, this would lead to skepticism.



Positive Introspection:

$$(\mathbf{4}) \quad \mathsf{K}A \rightarrow \mathsf{KK}A$$

Negative Introspection:

$$(\mathbf{5}) \quad MA \rightarrow \mathsf{K}MA$$

(5) corresponds to euclidity. Euclidity and reflexivity entail symmetry. So philosophers mostly reject (5).

(4) corresponds to transitivity. It is controversial.

Multi-Modal Logic

Multi-Modal Logic

If we want to talk about several agents, we need a multi-modal logic.

Definition

A **multi-modal Kripke model** consists of

- a non-empty set W,
- a set of binary relation $R_1, R_2, \ldots, R_n$ on W, and
- a function V that assigns to each sentence letter of $\mathfrak{L}_M$ a subset of W.

In epistemic logic, v is R_i-accessible from w iff v is compatible with the information agent i has at world w.

Multi-Modal Logic

The **language of multi-modal propositional logic** has several boxes $\Box_1, \Box_2, \ldots, \Box_n$ and diamonds $\Diamond_1, \Diamond_2, \ldots \Diamond_n$.

$$
\begin{array}{ll}
M, w \models \Box_i A & \text{iff } M, v \models A \text{ for all } v \text{ such that } wR_iv. \\
M, w \models \Diamond_i A & \text{iff } M, v \models A \text{ for some } v \text{ such that } wR_iv.
\end{array}
$$

Multi-Modal Logic

As before, we write the boxes as 'K' and the diamonds as 'M'.

- $\mathrm{M}_1\, p$
- $\mathrm{K}_1\, p$
- $\mathrm{K}_1 \mathrm{M}_2\, p$
- $\mathrm{K}_1\, p \rightarrow \mathrm{M}_2\, p$
- $\mathrm{K}_1\, p \rightarrow \mathrm{K}_2 \mathrm{K}_1\, p$

Interaction principles

Interaction principles

In multi-modal logics, we can impose constraints on individual accessibility relations:

- R_1 is reflexive
- R_2 is transitive
- etc.

but also on how different relations interact:

- if wR_1v then wR_2v
- if wR_1v then vR_2w
- if wR_1v and vR_2u then wR_3u
- etc.

Interaction principles

Constraints on the interaction between accessibility relations correspond to **interaction schemas** that link different operators.

$\Diamond_1 A \rightarrow \Diamond_2 A$

$\Diamond_1 A \rightarrow \Box_2 \Diamond_1 A$

etc.

Interaction principles

An interaction principle for multi-agent knowledge:

$$\mathrm{K}_1\mathrm{K}_2 A \rightarrow \mathrm{K}_1 A$$

But this follows from the (T)-schema for K_2:

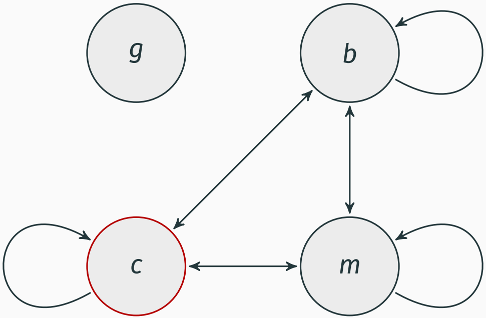
1.	$\mathrm{K}_2 A \rightarrow A$	(T)
2.	$\mathrm{K}_1(\mathrm{K}_2 A \rightarrow A)$	(1, Nec)
3.	$\mathrm{K}_1(\mathrm{K}_2 A \rightarrow A) \rightarrow (\mathrm{K}_1\mathrm{K}_2 A \rightarrow \mathrm{K}_1)$	(K)
4.	$\mathrm{K}_1\mathrm{K}_2 A \rightarrow \mathrm{K}_1 A$	(2, 3, MP)

Knowledge and Belief

Knowledge and Belief

A belief state represents the world as being a certain way.

We can ask, for every possible world, whether it matches what an agent believes.



Is the **doxastic accessibility relation**

- reflexive ($\Box A \to A$)?
- serial ($\Box A \to \Diamond A$)?
- symmetric ($A \to \Box\Diamond A$)?
- transitive ($\Box A \to \Box\Box A$)?
- euclidean ($\Diamond A \to \Diamond\Box A$)?

If we accept seriality, transitivity, and euclidity, we get the logic KD45.

$$M, w \models \mathsf{K}A \text{ iff } M, v \models A \text{ for all } v \text{ such that } wR_K v$$

$$M, w \models \mathsf{B}A \text{ iff } M, v \models A \text{ for all } v \text{ such that } wR_B v$$

A plausible interaction principle: $\mathsf{K}A \rightarrow \mathsf{B}A$

What does this mean for R_B and R_K?

$$\mathsf{K}A \rightarrow \mathsf{B}A$$

$$\Diamond_B A \rightarrow \Diamond_K A$$

Whenever $wR_B v$ then $wR_K v$.

Candidate Interaction Principles for B and K:

$$(\mathbf{KB})\quad \mathsf{K}A \rightarrow \mathsf{B}A$$

$$(\mathbf{PI})\quad \mathsf{B}A \rightarrow \mathsf{K}\mathsf{B}A$$

$$(\mathbf{NI})\quad \neg\mathsf{B}A \rightarrow \mathsf{K}\neg\mathsf{B}A$$

$$(\mathbf{SB})\quad \mathsf{B}A \rightarrow \mathsf{B}\mathsf{K}A$$

These entail

$$(\mathbf{B4})\quad \mathsf{B}A \rightarrow \mathsf{B}\mathsf{B}A$$

$$(\mathbf{B5})\quad \neg\mathsf{B}A \rightarrow \mathsf{B}\neg\mathsf{B}A$$

$$(\mathbf{KG})\quad \mathsf{M}\mathsf{K}A \rightarrow \mathsf{K}\mathsf{M}A$$

Knowledge and Possibility

Knowledge and Possibility

Let $\Diamond A$ mean that A is possible, in some circumstantial sense.

$$M, w \models \mathrm{K}A \text{ iff } M, v \models A \text{ for all } v \text{ such that } wR_K v$$

$$M, w \models \Diamond A \text{ iff } M, v \models A \text{ for some } v \text{ such that } wR_C v$$

The verificationist **principle of knowability**: $A \rightarrow \Diamond \mathrm{K} A$

1. Let p be any unknown truth. So $p \wedge \neg \mathrm{K} p$.
2. By the knowability principle, $\Diamond \mathrm{K}(p \wedge \neg \mathrm{K} p)$.
3. $\mathrm{K}(p \wedge \neg \mathrm{K} p)$ entails $\mathrm{K} p \wedge \mathrm{K} \neg \mathrm{K} p$.
4. $\mathrm{K} \neg \mathrm{K} p$ entails $\neg \mathrm{K} p$.
5. So $\mathrm{K} p$ and $\neg \mathrm{K} p$.