Logic 2: Modal Logic

Lecture 10

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Review

Review

We want to read the box as '*S* knows that'.

Possible-worlds analysis of knowledge

K*A* is true at *w* iff *A* is true at all worlds accessible from *w*.

What does accessibility mean here?

- *v* is accessible from *w* iff *v* is compatibility with what *S* knows at *w*.
- *v* is accessible from *w* iff *S* at *w* can't rule out *v*.
- *v* is accessible from *w* iff everything *S* knows at *w* is true at *v*.
- *v* is accessible from *w* iff *S* has the same experiences at *v* as at *w*?

Review

The **logic** of knowledge only depends on formal properties of the accessibility relation.

Review

Is the accessibility relation for knowledge reflexive?

Equivalently, is the (T)-schema valid in the logic of knowledge?

$$(\mathbf{T}) \quad \mathsf{K}A \rightarrow A$$

Plausibly, yes.

We then automatically get

$$(\mathbf{D}) \quad \mathsf{K}A \rightarrow \mathsf{M}A$$

Review

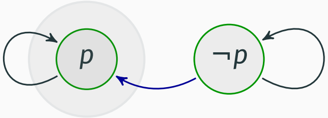
Should R be symmetric? Do we want (B) to come out valid?

$$(\mathbf{B})\quad A \rightarrow \mathrm{KM}\,A$$

Suppose you falsely believe $\neg p$.

- p is true.
- You believe that you know $\neg p$.
- You don't believe that you don't know $\neg p$.
- You don't know that you don't know $\neg p$.
- $\mathrm{K}\,\neg\mathrm{K}\,\neg p$ is false.
- $\mathrm{KM}\,p$ is false.

Also, this would lead to skepticism.



Review

Positive Introspection:

$$(\mathbf{4})\quad \mathsf{K}A \rightarrow \mathsf{KK}A$$

Negative Introspection:

$$(\mathbf{5})\quad MA \rightarrow \mathsf{K}MA$$

(5) corresponds to euclidity. Euclidity and reflexivity entail symmetry. So philosophers mostly reject (5).

(4) corresponds to transitivity. It is controversial.

Multi-Modal Logic

Multi-Modal Logic

If we want to talk about several agents, we need a multi-modal logic.

Definition

A **multi-modal Kripke model** consists of

- a non-empty set W,
- a set of binary relation $R_1, R_2, \ldots, R_n$ on W, and
- a function V that assigns to each sentence letter of $\mathfrak{L}_M$ a subset of W.

In epistemic logic, v is R_i-accessible from w iff v is compatible with the information agent i has at world w.

Multi-Modal Logic

The **language of multi-modal propositional logic** has several boxes $\Box_1, \Box_2, \ldots, \Box_n$ and diamonds $\Diamond_1, \Diamond_2, \ldots \Diamond_n$.

$$
\begin{aligned}
M, w \models \Box_i A \quad & \text{iff } M, v \models A \text{ for all } v \text{ such that } wR_iv. \\
M, w \models \Diamond_i A \quad & \text{iff } M, v \models A \text{ for some } v \text{ such that } wR_iv.
\end{aligned}
$$

Multi-Modal Logic

As before, we write the boxes as ‘K’ and the diamonds as ‘M’.

- $\mathrm{M}_1\, p$
- $\mathrm{K}_1\, p$
- $\mathrm{K}_1 \mathrm{M}_2\, p$
- $\mathrm{K}_1\, p \rightarrow \mathrm{M}_2\, p$
- $\mathrm{K}_1\, p \rightarrow \mathrm{K}_2 \mathrm{K}_1\, p$

Interaction principles

Interaction principles

In multi-modal logics, we can impose constraints on individual accessibility relations:

- R_1 is reflexive
- R_2 is transitive
- etc.

but also on how different relations interact:

- if wR_1v then wR_2v
- if wR_1v then vR_2w
- if wR_1v and vR_2u then wR_3u
- etc.

Interaction principles

Constraints on the interaction between accessibility relations correspond to **interaction schemas** that link different operators.

$\Diamond_1 A \rightarrow \Diamond_2 A$

$\Diamond_1 A \rightarrow \Box_2 \Diamond_1 A$

etc.

Interaction principles

An interaction principle for multi-agent knowledge:

$$K_1 K_2 A \rightarrow K_1 A$$

But this follows from the (T)-schema for K_2:

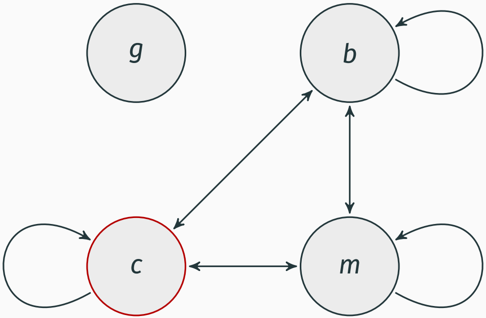
1.	$K_2 A \rightarrow A$	(T)
2.	$K_1(K_2 A \rightarrow A)$	(1, Nec)
3.	$K_1(K_2 A \rightarrow A) \rightarrow (K_1 K_2 A \rightarrow K_1 A)$	(K)
4.	$K_1 K_2 A \rightarrow K_1 A$	(2, 3, MP)

Knowledge and Belief

Knowledge and Belief

A belief state represents the world as being a certain way.

We can ask, for every possible world, whether it matches what an agent believes.



Is the **doxastic accessibility relation**

- reflexive ($\Box A \rightarrow A$)?
- serial ($\Box A \rightarrow \Diamond A$)?
- symmetric ($A \rightarrow \Box\Diamond A$)?
- transitive ($\Box A \rightarrow \Box\Box A$)?
- euclidean ($\Diamond A \rightarrow \Diamond\Box A$)?

If we accept seriality, transitivity, and euclidity, we get the logic KD45.

Knowledge and Belief

$$M, w \models \mathrm{K}A \text{ iff } M, v \models A \text{ for all } v \text{ such that } wR_K v$$

$$M, w \models \mathrm{B}A \text{ iff } M, v \models A \text{ for all } v \text{ such that } wR_B v$$

A plausible interaction principle: $\mathrm{K}A \to \mathrm{B}A$

What does this mean for R_B and R_K?

$$\mathrm{K}A \to \mathrm{B}A$$

$$\neg \mathrm{B}A \to \neg \mathrm{K}A$$

Whenever $\neg A$ is B-accessible, then $\neg A$ is K-accessible.

Whenever wR_Bv then wR_Kv.

Candidate Interaction Principles for B and K:

(**KB**) $\mathrm{K}A \rightarrow \mathrm{B}A$

(**PI**) $\mathrm{B}A \rightarrow \mathrm{K}\mathrm{B}A$

(**NI**) $\neg\mathrm{B}A \rightarrow \mathrm{K}\neg\mathrm{B}A$

(**SB**) $\mathrm{B}A \rightarrow \mathrm{B}\mathrm{K}A$

These entail

(**B4**) $\mathrm{B}A \rightarrow \mathrm{B}\mathrm{B}A$

(**B5**) $\neg\mathrm{B}A \rightarrow \mathrm{B}\neg\mathrm{B}A$

(**KG**) $\mathrm{M}\mathrm{K}A \rightarrow \mathrm{K}\mathrm{M}A$

Knowledge and Possibility

Knowledge and Possibility

Let $\Diamond A$ mean that A is possible, in some circumstantial sense.

$$M, w \models \mathsf{K}A \text{ iff } M, v \models A \text{ for all } v \text{ such that } wR_K v$$

$$M, w \models \Diamond A \text{ iff } M, v \models A \text{ for some } v \text{ such that } wR_C v$$

The verificationist **principle of knowability**: $A \rightarrow \Diamond \mathsf{K} A$

1. Let p be any unknown truth. So $p \wedge \neg \mathsf{K} p$.
2. By the knowability principle, $\Diamond \mathsf{K}(p \wedge \neg \mathsf{K} p)$.
3. $\mathsf{K}(p \wedge \neg \mathsf{K} p)$ entails $\mathsf{K} p \wedge \mathsf{K} \neg \mathsf{K} p$.
4. $\mathsf{K} \neg \mathsf{K} p$ entails $\neg \mathsf{K} p$.
5. So $\mathsf{K} p$ and $\neg \mathsf{K} p$.