Logic 2: Modal Logic

Lecture 6

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Review

Review

Kripke model

A Kripke model of $\mathfrak{L}_M$ is a triple $\langle W, R, V \rangle$ consisting of

- a non-empty set W,
- a binary relation R on W, and
- a function V that assigns to each sentence letter of $\mathfrak{L}_M$ a subset of W.

Kripke semantics

$\Box A$ is true at w iff A is true at all worlds accessible from w.

$\Diamond A$ is true at w iff A is true at some world accessible from w.

Review

K-valid

A sentence is K-valid iff it is true at all worlds in all Kripke models.

S5-valid

A sentence is S5-valid iff it is true at all worlds in all basic models.

Review

Schema		S5	K
(K)	$\Box(A \to B) \to (\Box A \to \Box B)$	✓	✓
(T)	$\Box A \to A$	✓	—
(D)	$\Box A \to \Diamond A$	✓	—
(B)	$A \to \Box\Diamond A$	✓	—
(4)	$\Box A \to \Box\Box A$	✓	—
(5)	$\Diamond A \to \Box\Diamond A$	✓	—
(G)	$\Diamond\Box A \to \Box\Diamond A$	✓	—

Tree rules for K

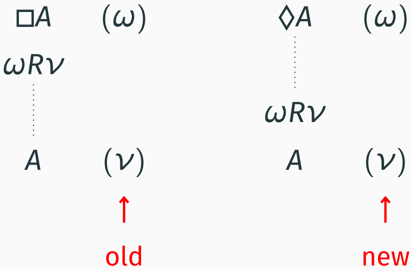
Tree rules for K

Tree rules for S5

$$\begin{array}{cc} \Box A & (\omega) \\ \vdots & \\ A & (\nu) \\ & \uparrow \\ & \text{old} \end{array} \qquad\qquad \begin{array}{cc} \Diamond A & (\omega) \\ \vdots & \\ A & (\nu) \\ & \uparrow \\ & \text{new} \end{array}$$

Tree rules for K

Tree rules for K



$\Box A \quad (\omega)$

$\omega R \nu$

$\vdots$

$A \quad (\nu)$

↑ old

$\Diamond A \quad (\omega)$

$\vdots$

$\omega R \nu$

$A \quad (\nu)$

↑ new

Tree rules for K

Target sentence: $(\Box p \wedge \Diamond q) \rightarrow \Diamond p$

1. $\neg((\Box p \wedge \Diamond q) \rightarrow \Diamond p)$ (w) (Ass.)

Tree rules for K

Target sentence: $(\Box p \wedge \Diamond q) \rightarrow \Diamond p$

1. $\neg((\Box p \wedge \Diamond q) \rightarrow \Diamond p)$ (w) (Ass.)
2. $\Box p \wedge \Diamond q$ (w) (1)
3. $\neg \Diamond p$ (w) (1)

Tree rules for K

Target sentence: $(\Box p \wedge \Diamond q) \rightarrow \Diamond p$

1.	$\neg((\Box p \wedge \Diamond q) \rightarrow \Diamond p)$	(w)	(Ass.)
2.	$\Box p \wedge \Diamond q$	(w)	(1)
3.	$\neg \Diamond p$	(w)	(1)
4.	$\Box p$	(w)	(2)
5.	$\Diamond q$	(w)	(2)

Tree rules for K

Target sentence: $(\Box p \wedge \Diamond q) \rightarrow \Diamond p$

1.	$\neg((\Box p \wedge \Diamond q) \rightarrow \Diamond p)$	(w)	(Ass.)
2.	$\Box p \wedge \Diamond q$	(w)	(1)
3.	$\neg \Diamond p$	(w)	(1)
4.	$\Box p$	(w)	(2)
5.	$\Diamond q$	(w)	(2)
6.	wRv		(5)
7.	q	(v)	(5)

Tree rules for K

Target sentence: $(\Box p \wedge \Diamond q) \rightarrow \Diamond p$

1.	$\neg((\Box p \wedge \Diamond q) \rightarrow \Diamond p)$	(w)	(Ass.)
2.	$\Box p \wedge \Diamond q$	(w)	(1)
3.	$\neg \Diamond p$	(w)	(1)
4.	$\Box p$	(w)	(2)
5.	$\Diamond q$	(w)	(2)
6.	wRv		(5)
7.	q	(v)	(5)
8.	p	(v)	(4,6)

Tree rules for K

Target sentence: $(\Box p \wedge \Diamond q) \rightarrow \Diamond p$

1.	$\neg((\Box p \wedge \Diamond q) \rightarrow \Diamond p)$	(w)	(Ass.)
2.	$\Box p \wedge \Diamond q$	(w)	(1)
3.	$\neg\Diamond p$	(w)	(1)
4.	$\Box p$	(w)	(2)
5.	$\Diamond q$	(w)	(2)
6.	wRv		(5)
7.	q	(v)	(5)
8.	p	(v)	(4,6)
8.	$\neg p$	(v)	(3,6)
	x		

The system T

The system T

Let $\Box p$ mean 'you know that p'.

Which of these are valid?

- $\Box p \rightarrow p$
- $\Box p \rightarrow \Diamond p$
- $p \rightarrow \Box\Diamond p$

Suppose you falsely believe $\neg p$.

- p is true.
- You believe that you know $\neg p$.
- You don't believe that you don't know $\neg p$.
- You don't know that you don't know $\neg p$.
- $\Box\neg\Box\neg p$ is false.
- $\Box\Diamond p$ is false.

The system T

We want a logic of knowledge with (K), (T), and (D), but not (B).

Schema	S5	K
(K) $\Box(A \to B) \to (\Box A \to \Box B)$	✓	✓
(T) $\Box A \to A$	✓	—
(D) $\Box A \to \Diamond A$	✓	—
(B) $A \to \Box\Diamond A$	✓	—
(4) $\Box A \to \Box\Box A$	✓	—
(5) $\Diamond A \to \Box\Diamond A$	✓	—
(G) $\Diamond\Box A \to \Box\Diamond A$	✓	—

Schema	S5	K	T
(K) $\Box(A \to B) \to (\Box A \to \Box B)$	✓	✓	✓

The system T

We want a logic of knowledge with (K), (T), and (D), but not (B).

(T) $\Box A \rightarrow A$

$\Box A \rightarrow A$ is false at a world w in a Kripke model iff

- A is false at w
- $\Box A$ is true at w
- A is true at all worlds accessible from w

This can't happen if w is accessible from itself.

The system T

Definition: K-valid

A sentence A is **K-valid** (for short, $\models_K A$) iff A is true at every world in every Kripke model.

Definition: T-valid

A sentence A is **T-valid** (for short, $\models_T A$) iff A is true at every world in every Kripke model in which each world has access to itself.

The set of all T-valid sentences is **system T**.

The system T

A relation R on a set W is called **reflexive** if every member of W is R-related to itself.

Definition: T-valid

A sentence A is **T-valid** (for short, $\models_T A$) iff A is true at every world in every Kripke model in which each world has access to itself.

Definition: T-valid

A sentence A is **T-valid** (for short, $\models_T A$) iff A is true at every world in every Kripke model with a reflexive accessibility relation.

The system T

Schema		S5	K	T
(K)	$\Box(A \to B) \to (\Box A \to \Box B)$	✓	✓	✓
(T)	$\Box A \to A$	✓	—	✓
(D)	$\Box A \to \Diamond A$	✓	—	✓
(B)	$A \to \Box\Diamond A$	✓	—	—
(4)	$\Box A \to \Box\Box A$	✓	—	—
(5)	$\Diamond A \to \Box\Diamond A$	✓	—	—
(G)	$\Diamond\Box A \to \Box\Diamond A$	✓	—	—

The system T

New tree rule for system T:

⋮

$\omega R \omega$

↑

old

The system T

Target sentence: $\Box p \to p$

1.	$\neg(\Box p \to p)$	(w)	(Ass.)
2.	$\Box p$	(w)	(1)
3.	$\neg p$	(w)	(1)

The system T

Target sentence: $\Box p \to p$

1.	$\neg(\Box p \to p)$	(w)	(Ass.)
2.	$\Box p$	(w)	(1)
3.	$\neg p$	(w)	(1)
4.	wRw		(Ref.)
5.	p	(w)	(2,4)
	x		

The system S4

The system S4

Let $\Box p$ mean 'you know that p'.

- $\Box p \rightarrow p$ is plausibly valid.
- $\Box p \rightarrow \Diamond p$ is plausibly valid.
- $p \rightarrow \Box\Diamond p$ is not.
- What about $\Box p \rightarrow \Box\Box p$?

The system S4

Schema		S5	K	T	S4
(K)	$\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$	✓	✓	✓	✓
(T)	$\Box A \rightarrow A$	✓	—	✓	✓
(D)	$\Box A \rightarrow \Diamond A$	✓	—	✓	✓
(B)	$A \rightarrow \Box\Diamond A$	✓	—	—	—
(4)	$\Box A \rightarrow \Box\Box A$	✓	—	—	✓
(5)	$\Diamond A \rightarrow \Box\Diamond A$	✓	—	—	—
(G)	$\Diamond\Box A \rightarrow \Box\Diamond A$	✓	—	—	—

The system S4

How can we define a concept of validity that makes (4) valid but not (B)?

(4) $\Box A \rightarrow \Box\Box A$

$\Box A \rightarrow \Box\Box A$ is false at a world *w* in a Kripke model iff

- $\Box A$ is true at *w*
- $\Box\Box A$ is false at *w*
- Wherever you go in one step from *w*, *A* is true.
- It is not the case that wherever you go in two steps from *w*, *A* is true.

This can't happen if any world that can be reached in two steps can be reached in one step.

The system S4

Definition: S4-valid

A sentence A is **S4-valid** (for short, $\models_{S4} A$) iff A is true at every world in every Kripke model in which

(a) each world has access to itself, and

(b) if a world w has access to some world v, and v has access to a world u, then w has access to u.

The set of all S4-valid sentences is **system S4**.

The system S4

A relation R on a set W is called **transitive** if whenever wRv and vRu, then wRu.

Definition: S4-valid

A sentence A is **S4-valid** (for short, $\models_{S4} A$) iff A is true at every world in every Kripke model in which

(a) each world has access to itself, and

(b) if a world w has access to some world v, and v has access to a world u, then w has access to u.

Definition: S4-valid

A sentence A is **S4-valid** (for short, $\models_{S4} A$) iff A is true at every world in every Kripke model whose accessibility relation is reflexive and transitive.

The system S4

Schema		S5	K	T	S4
(K)	$\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$	✓	✓	✓	✓
(T)	$\Box A \rightarrow A$	✓	—	✓	✓
(D)	$\Box A \rightarrow \Diamond A$	✓	—	✓	✓
(B)	$A \rightarrow \Box\Diamond A$	✓	—	—	—
(4)	$\Box A \rightarrow \Box\Box A$	✓	—	—	✓
(5)	$\Diamond A \rightarrow \Box\Diamond A$	✓	—	—	—
(G)	$\Diamond\Box A \rightarrow \Box\Diamond A$	✓	—	—	—

The system S4

New tree rule for system S4:

$$\begin{array}{c} \omega R \nu \\ \nu R \upsilon \\ \vdots \\ \omega R \upsilon \end{array}$$

The system S4

Target sentence: $\Box A \to \Box\Box A$

1.	$\neg(\Box A \to \Box\Box A)$	(w)	(Ass.)
2.	$\Box A$	(w)	(1)
3.	$\neg\Box\Box A$	(w)	(1)
4.	wRv		(3)
5.	$\neg\Box A$	(v)	(3)
6.	vRu		(5)
7.	$\neg A$	(u)	(5)
8.	wRu		(4,6,Tr.)
5.	A	(u)	(2,8)
	X		

Further systems

Further systems

Definition: K4-valid

A sentence A is **K4-valid** (for short, $\models_{K4} A$) iff A is true at every world in every Kripke model whose accessibility relation is transitive.

Further systems

Schema		S5	K	T	S4	K4
(K)	$\Box(A \to B) \to (\Box A \to \Box B)$	✓	✓	✓	✓	✓
(T)	$\Box A \to A$	✓	—	✓	✓	—
(D)	$\Box A \to \Diamond A$	✓	—	✓	✓	—
(B)	$A \to \Box\Diamond A$	✓	—	—	—	—
(4)	$\Box A \to \Box\Box A$	✓	—	—	✓	✓
(5)	$\Diamond A \to \Box\Diamond A$	✓	—	—	—	—
(G)	$\Diamond\Box A \to \Box\Diamond A$	✓	—	—	—	—

Further systems

Can you think of a requirement on the accessibility relation that makes $\Box A \to \Diamond A$ valid?

A relation R on a set W is called **serial** if every member of W is R-related to some member of W. (“No dead ends”)

Definition: D-valid

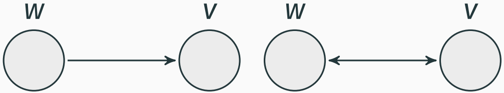
A sentence A is **D-valid** (for short, $\models_D A$) iff A is true at every world in every Kripke model whose accessibility relation is serial.

Further systems

Schema		S5	K	T	S4	K4	D
(K)	$\Box(A \to B) \to (\Box A \to \Box B)$	✓	✓	✓	✓	✓	✓
(T)	$\Box A \to A$	✓	—	✓	✓	—	—
(D)	$\Box A \to \Diamond A$	✓	—	✓	✓	—	✓
(B)	$A \to \Box\Diamond A$	✓	—	—	—	—	—
(4)	$\Box A \to \Box\Box A$	✓	—	—	✓	✓	—
(5)	$\Diamond A \to \Box\Diamond A$	✓	—	—	—	—	—
(G)	$\Diamond\Box A \to \Box\Diamond A$	✓	—	—	—	—	—

Further systems

A relation R on a set W is called **symmetric** if whenever wRv then vRw.



Further systems

A relation R on a set W is called **symmetric** if whenever wRv then vRw.

Definition: B-valid

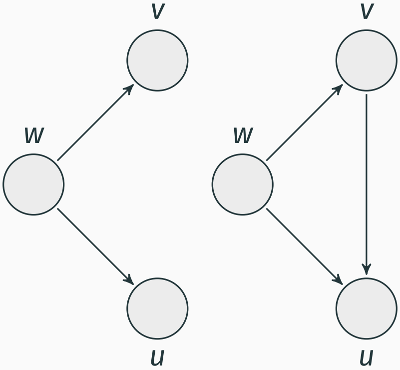
A sentence A is **B-valid** (for short, $\models_B A$) iff A is true at every world in every Kripke model whose accessibility relation is reflexive and symmetric.

Further systems

Schema		S5	K	T	S4	K4	D	B
(K)	$\Box(A \to B) \to (\Box A \to \Box B)$	✓	✓	✓	✓	✓	✓	✓
(T)	$\Box A \to A$	✓	—	✓	✓	—	—	✓
(D)	$\Box A \to \Diamond A$	✓	—	✓	✓	—	✓	✓
(B)	$A \to \Box\Diamond A$	✓	—	—	—	—	—	✓
(4)	$\Box A \to \Box\Box A$	✓	—	—	✓	✓	—	—
(5)	$\Diamond A \to \Box\Diamond A$	✓	—	—	—	—	—	—
(G)	$\Diamond\Box A \to \Box\Diamond A$	✓	—	—	—	—	—	—

Further systems

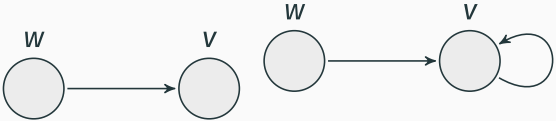
A relation R on a set W is called **euclidean** if whenever wRv and wRu, then vRu.



Further systems

A relation R is called **euclidean** if whenever wRv and wRu, then vRu.

Note: w, v, and u need not be distinct worlds! $\forall x \forall y \forall z(xRy \land xRz \rightarrow yRz)$.



Further systems

A relation R on a set W is called **euclidean** if whenever wRv and wRu, then vRu.

Definition

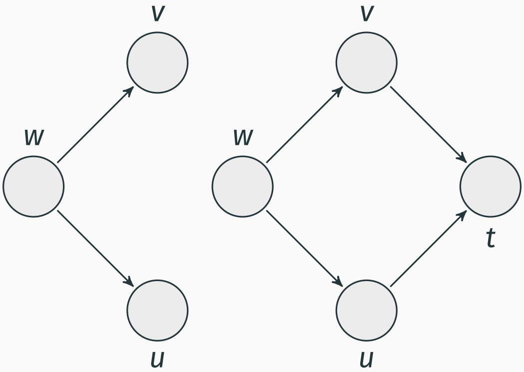
A sentence A is **K45-valid** (for short, $\models_{K45} A$) iff A is true at every world in every Kripke model whose accessibility relation is transitive and euclidean.

Further systems

Schema		S5	K	T	S4	K4	D	B	K45
(K)	$\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$	✓	✓	✓	✓	✓	✓	✓	✓
(T)	$\Box A \rightarrow A$	✓	—	✓	✓	—	—	✓	—
(D)	$\Box A \rightarrow \Diamond A$	✓	—	✓	✓	—	✓	✓	—
(B)	$A \rightarrow \Box\Diamond A$	✓	—	—	—	—	—	✓	—
(4)	$\Box A \rightarrow \Box\Box A$	✓	—	—	✓	✓	—	—	✓
(5)	$\Diamond A \rightarrow \Box\Diamond A$	✓	—	—	—	—	—	—	✓
(G)	$\Diamond\Box A \rightarrow \Box\Diamond A$	✓	—	—	—	—	—	—	—

Further systems

A relation R on a set W is called **convergent** if whenever wRv and wRu, then there is some t such that wRt and vRt.



Further systems

A relation R on a set W is called **convergent** if whenever wRv and wRu, then there is some t such that wRt and vRt.

Definition

A sentence A is **S4.2-valid** (for short, $\models_{S4.2} A$) iff A is true at every world in every Kripke model whose accessibility relation is reflexive, transitive, and convergent.

Further systems

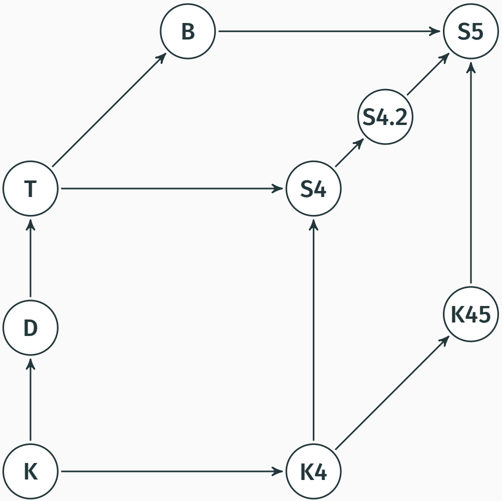
Schema		S5	K	T	S4	K4	D	B	K45	S4.2
(K)	$\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$	✓	✓	✓	✓	✓	✓	✓	✓	✓
(T)	$\Box A \rightarrow A$	✓	—	✓	✓	—	—	✓	—	✓
(D)	$\Box A \rightarrow \Diamond A$	✓	—	✓	✓	—	✓	✓	—	✓
(B)	$A \rightarrow \Box\Diamond A$	✓	—	—	—	—	—	✓	—	—
(4)	$\Box A \rightarrow \Box\Box A$	✓	—	—	✓	✓	—	—	✓	✓
(5)	$\Diamond A \rightarrow \Box\Diamond A$	✓	—	—	—	—	—	—	✓	—
(G)	$\Diamond\Box A \rightarrow \Box\Diamond A$	✓	—	—	—	—	—	—	—	✓

Further systems

Schema		Condition On R
(T)	$\Box A \rightarrow A$	R is reflexive: every world in W is accessible from itself
(D)	$\Box A \rightarrow \Diamond A$	R is serial: every world in W can access some world in W
(B)	$A \rightarrow \Box\Diamond A$	R is symmetric: whenever wRv then vRw
(4)	$A \rightarrow \Box\Box A$	R is transitive: whenever wRv and vRu, then wRu
(5)	$\Diamond A \rightarrow \Box\Diamond A$	R is euclidean: whenever wRv and wRu, then vRu
(G)	$\Diamond\Box A \rightarrow \Box\Diamond A$	R is convergent: whenever wRv and wRu, then there is some t such that vRt and uRt

Further systems

K	–
T	R is reflexive
D	R is serial
K4	R is transitive
K45	R is transitive and euclidean
B	R is reflexive and symmetric
S4	R is reflexive and transitive
S4.2	R is reflexive, transitive, and convergent
S5	R is reflexive, transitive, and symmetric
S5	R is universal



Further systems