Logic 2: Modal Logic

Lecture 5

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Recap

Recap

- We want to formalize reasoning about possibility, obligation, knowledge, past and future, etc.
- We have added new sentence operators $\Box$ and $\Diamond$ to the language of propositional logic.
- We have interpreted these as quantifiers over possible worlds.

Recap

- A basic model of $\mathfrak{L}_M$ consists of a non-empty set W and an interpretation function V that assigns truth-values to sentence letters at members of W.
- $\Box A$ is true at w in $\langle W, V\rangle$ iff A is true at all $v \in W$.
- $\Diamond A$ is true at w in $\langle W, V\rangle$ iff A is true at some $v \in W$.
- A sentence is valid if it is true at all worlds in all models.

Recap

All instances of the following schemas come out valid.

(**K**) $\Box(A \to B) \to (\Box A \to \Box B)$

(**T**) $\Box A \to A$

(**D**) $\Box A \to \Diamond A$

(**B**) $A \to \Box\Diamond A$

(**4**) $\Box A \to \Box\Box A$

(**5**) $\Diamond A \to \Box\Diamond A$

(**G**) $\Diamond\Box A \to \Box\Diamond A$

We don’t always want these to be valid.

Recap

What is possible often depends on what is actual.

- You can travel from Auckland to Sydney by train.
- Bob might be in his office.
- You may keep the library books for one more week.

Recap

Revised possible-worlds analysis

$\Diamond p$ is true at w iff p is true at a world that's possible relative to p.

Revised possible-worlds analysis

$\Diamond p$ is true at w iff p is true at a world that's possible relative to p.
$\Box p$ is true at w iff p is true at all worlds that are possible relative to p.

(Montague 1955, Meredith and Prior 1956, Kanger 1957, Hintikka 1957, Montague 1960, Hintikka 1960, Hintikka 1961, Prior 1962, **Kripke 1963**.)

Kripke models

Kripke models

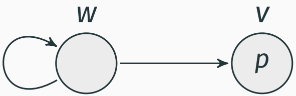
Definition: Kripke model

A Kripke model of $\mathfrak{L}_M$ is a triple $\langle W, R, V\rangle$ consisting of

- a non-empty set W (the "worlds"),
- a binary ("accessibility") relation R on W, and
- a function V that assigns to each sentence letter of $\mathfrak{L}_M$ a subset of W.

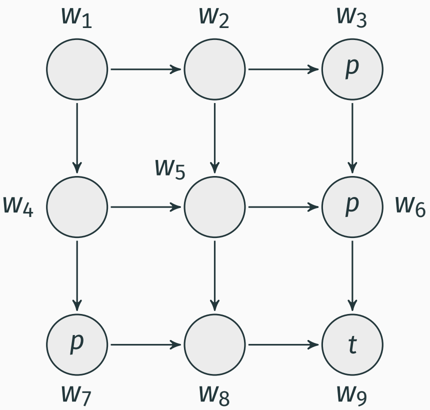
Intuitively: wRv iff v is possible relative to w.

Kripke models



- $\Diamond p$ is true at w.
- $\Diamond p$ is false at v.
- $\Box p$ is false at w.
- $\Box p$ is true at v. (Think $\neg\Diamond\neg p$.)
- $\Box\neg p$ is true at v. (Think $\neg\Diamond p$.)
- $\Box\Diamond p$ is false at w.
- $\Box\Diamond p$ is true at v.

Kripke models



Where is $\Diamond t$ true? Where is $\Diamond\Box t$ true? Where is $\Diamond p$ true? Where is $\Box\Diamond p$ true? Can you find a sentence that is true only at w_1?

The system K

The system K

Definition: K-valid

A sentence A is **K-valid** (for short, $\models_K A$) iff A is true at every world in every Kripke model.

Definition: S5-valid

A sentence A is **S5-valid** (for short, $\models_{S5} A$) iff A is true at every world in every basic model.

The set of all K-valid sentences is **system K**.

The set of all S5-valid sentences is **system S5**.

The system K

An axiomatization of S5:

(**Dual**) $\neg\Diamond A \leftrightarrow \Box\neg A$

(**K**) $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

(**T**) $\Box A \rightarrow A$

(**4**) $\Box A \rightarrow \Box\Box A$

(**5**) $\Diamond A \rightarrow \Box\Diamond A$

(**CPL**) Any truth-functional consequence of sentences in the system is in the system.

(**Nec**) If A is in the system then so is $\Box A$.

The system K

An axiomatization of K:

(**Dual**) $\neg\Diamond A \leftrightarrow \Box\neg A$

(**K**) $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

(**CPL**) Any truth-functional consequence of sentences in the system is in the system.

(**Nec**) If A is in the system then so is $\Box A$.

The system K

Distinguish:

Schema (K): $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

System K: The set of sentences that are true at all worlds in all Kripke models

System K contains many sentences that aren't instances of (K).

- $p \vee \neg p$
- $\Box p \rightarrow \neg\Diamond\neg p$
- $\Box p \rightarrow \Box(p \vee q)$

The system K

Schema		S5	K
(K)	$\Box(A \to B) \to (\Box A \to \Box B)$	✓	✓
(T)	$\Box A \to A$	✓	—
(D)	$\Box A \to \Diamond A$	✓	—
(B)	$A \to \Box\Diamond A$	✓	—
(4)	$\Box A \to \Box\Box A$	✓	—
(5)	$\Diamond A \to \Box\Diamond A$	✓	—
(G)	$\Diamond\Box A \to \Box\Diamond A$	✓	—

Tree rules for K

Tree rules for S5

$$\begin{array}{ll} \Box A & (\omega) \\ \vdots & \\ A & (\nu) \\ & \uparrow \\ & \text{old} \end{array} \qquad \begin{array}{ll} \Diamond A & (\omega) \\ \vdots & \\ A & (\nu) \\ & \uparrow \\ & \text{new} \end{array}$$

Tree rules for K

$$\begin{array}{ll} \Box A & (\omega) \\ \omega R \nu & \\ \vdots & \\ A & (\nu) \\ & \uparrow \\ & \text{old} \end{array} \qquad\qquad \begin{array}{ll} \Diamond A & (\omega) \\ \vdots & \\ \omega R \nu & \\ A & (\nu) \\ & \uparrow \\ & \text{new} \end{array}$$

Target sentence: $(\Box p \wedge \Diamond q) \rightarrow \Diamond p$

1. $\neg((\Box p \wedge \Diamond q) \rightarrow \Diamond p)$ (w) (Ass.)

Tree rules for K

Target sentence: $(\Box p \wedge \Diamond q) \rightarrow \Diamond p$

1.	$\neg((\Box p \wedge \Diamond q) \rightarrow \Diamond p)$	(w)	(Ass.)
2.	$\Box p \wedge \Diamond q$	(w)	(1)
3.	$\neg \Diamond p$	(w)	(1)

Tree rules for K

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3.	$\neg \Diamond p$	(w)	(1)
4.	$\Box p$	(w)	(2)
5.	$\Diamond q$	(w)	(2)

Tree rules for K

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2.	$\Box p \wedge \Diamond q$	(w)	(1)
3.	$\neg \Diamond p$	(w)	(1)
4.	$\Box p$	(w)	(2)
5.	$\Diamond q$	(w)	(2)
6.	wRv		(5)
7.	q	(v)	(5)

Tree rules for K

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6.	wRv		(5)
7.	q	(v)	(5)
8.	p	(v)	(4,6)

Tree rules for K

Target sentence: $(\Box p \wedge \Diamond q) \rightarrow \Diamond p$

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4.	$\Box p$	(w)	(2)
5.	$\Diamond q$	(w)	(2)
6.	wRv		(5)
7.	q	(v)	(5)
8.	p	(v)	(4,6)
8.	$\neg p$ x	(v)	(3,6)