Logic 2: Modal Logic

Lecture 3

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Two paths to logic

Two paths to logic

- We have added the box $\Box$ and the diamond $\Diamond$ to the language of propositional logic.
- $\Box$ and $\Diamond$ have different meanings in different applications of modal logic.
- Often, the box stands for some kind of necessity and the diamond for some kind of possibility.
- We always assume that the box and the diamond are duals, so that $\neg\Box A$ is equivalent to $\Diamond\neg A$ and $\neg\Diamond A$ is equivalent to $\Box\neg A$.

Two paths to logic

A **proof-theoretic** approach to logic would now define formal rules for reasoning in $\mathfrak{L}_M$.

- From $\Box(A \wedge B)$ one may infer $\Box A$.
- From $\Box A$ one may infer $\Diamond A$.
- From A one may infer $\Diamond A$.
- ...

But:

How do we know these rules are correct?

How do we know we haven't missed any rules?

Some candidate rules are hard to evaluate intuitively.

- From $\Diamond(\Diamond\Diamond A \to \Box\Diamond B)$ one may infer $\Diamond\Diamond A \to \Box\Diamond B$. (Georgacarakos 1978)

Two paths to logic

A **model-theoretic** approach to logic begins by formalizing the concepts of validity and entailment.

- An $\mathfrak{L}_M$-sentence A is valid (for short, $\models A$) if A is true in every conceivable scenario, under every interpretation of the sentence letters.
- Some $\mathfrak{L}_M$-sentences $A_1, A_2, \ldots$ entail a sentence B (for short $A_1, A_2, \ldots \models B$ if there is no conceivable scenario and interpretation (of the sentence letters) that makes $A_1, A_2, \ldots$ true and B false.
- $A \models B$ iff $\models A \to B$.

Two paths to logic

We need to explain:

- What is a scenario? What is an interpretation of the sentence letters?
- When is an $\mathfrak{L}_M$-sentence true at a scenario under an interpretation?

Answering these questions will allow us to figure out whether

- A entails $\Diamond A$,
- $\Diamond(\Diamond\Diamond A \to \Box\Diamond B)$ entails $\Diamond\Diamond A \to \Box\Diamond B$,
- ...

Possible-worlds semantics

Possible-worlds semantics

A Leibnizian idea:

- Our world is one of many possible worlds.
- A sentence can be true at some worlds and false at others.
- ‘It is necessary that *p*’ says that *p* is true at all worlds.
- ‘It is possible that *p*’ says that *p* is true at some worlds.

Possible-worlds semantics

$\Box A$ is true (at a world) iff A is true at all worlds.
$\Diamond A$ is true (at a world) iff A is true at some world.

You can check:

$\Box\neg A$ is true iff $\neg\Diamond A$ is true.

$\Diamond\neg A$ is true iff $\neg\Box A$ is true.

Possible-worlds semantics

$\Box A$ is true (at a world) iff A is true at all worlds.
$\Diamond A$ is true (at a world) iff A is true at some world.

Is $\Diamond p \to \Box p$ valid?

- What would a scenario and interpretation have to look like for $\Diamond p \to \Box p$ to be false?
- $\Diamond p$ would have to be true and $\Box p$ false.
- p would have to be true at some world and not at all worlds.
- This is easily conceivable.

Possible-worlds semantics

$\Box A$ is true (at a world) iff A is true at all worlds.
$\Diamond A$ is true (at a world) iff A is true at some world.

Is $p \to \Diamond p$ valid?

Yes.

- Suppose $p \to \Diamond p$ is false in some scenario under some interpretation of p.
- Then p is true and $\Diamond p$ is false (in that scenario under that interpretation).
- Then p is true (in the actual world) but p is false at all worlds (in that scenario...).
- This is impossible.

Models

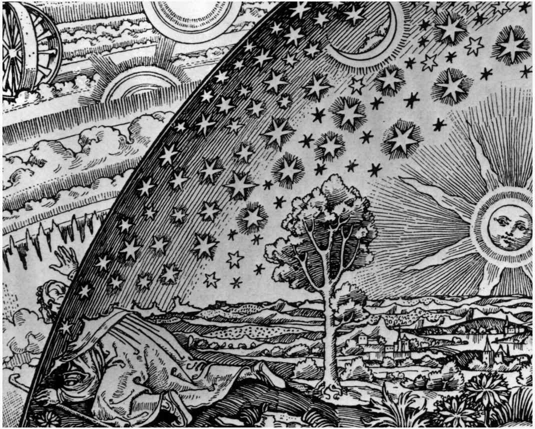
Models

$$\models A \quad \Leftrightarrow \quad A \text{ is valid}$$

$$\Leftrightarrow \quad A \text{ is true in every conceivable scenario, on any interpretation of the non-logical expressions.}$$

Models

A scenario:

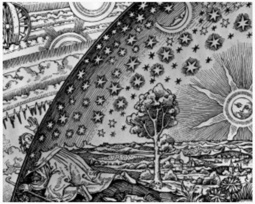


An interpretation of the sentence letters:

p : The sun is shining.
q : It is cloudy.
r : There is at least one tree.
s : Someone is wearing trousers.
t : There are many stars.
...

Models

A scenario:



An interpretation of the sentence letters:

$p \Leftrightarrow$ The sun is shining.
$q \Leftrightarrow$ It is cloudy.
$r \Leftrightarrow$ There is at least one tree.
$s \Leftrightarrow$ Someone is wearing trousers.
$t \Leftrightarrow$ There are many stars.
⋮

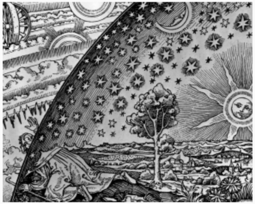
p is true.
q is false.
r is true.
s is false.
t is true.
...

$p \wedge q$ is false.
$p \vee q$ is true.
$p \rightarrow (s \rightarrow (q \vee r))$ is true.
...

Models

A model for the language of classical propositional logic:

A scenario:



An interpretation of the sentence letters:

$p \Leftrightarrow$ The sun is shining.
$q \Leftrightarrow$ It is cloudy.
$r \Leftrightarrow$ There is at least one tree.
$s \Leftrightarrow$ Someone is wearing trousers.
$t \Leftrightarrow$ There are many stars.
$\vdots$

p is true.
q is false.
r is true.
s is false.
t is true.
...

$p \wedge q$ is false.
$p \vee q$ is true.
$p \rightarrow (s \rightarrow (q \vee r))$ is true.
...

Models

A **model** for a language is a partial specification of

- a conceivable scenario and
- an interpretation of the language's non-logical expressions

that contains just enough information to determine the truth-value of all sentences.

$$\models A \quad \Leftrightarrow \quad A \text{ is valid}$$

$$\Leftrightarrow \quad A \text{ is true in every conceivable scenario, on any interpretation of the non-logical expressions.}$$

$$\Leftrightarrow \quad A \text{ is true in every model.}$$

Models

In classical propositional logic,

$$\begin{aligned} \models A \quad &\Leftrightarrow \quad A \text{ is valid} \\ &\Leftrightarrow \quad A \text{ is true in every model} \\ &\Leftrightarrow \quad A \text{ is true under every assignment of truth-values to the sentence letters.} \end{aligned}$$

Models

A complete scenario for $\mathfrak{L}_M$ would specify how many worlds there are, and what happens at each of them.

Together with an interpretation of the sentence letters, this would allow us to figure out the truth-value of all $\mathfrak{L}_M$-sentences (at all worlds).

A model for $\mathfrak{L}_M$ specifies

- how many worlds there are,
- which sentence letters are true at which worlds.

This is enough to figure out the truth-value of all $\mathfrak{L}_M$-sentences (at all worlds).

Formal possible-worlds semantics

Formal possible-worlds semantics

A **(basic) model** for $\mathfrak{L}_M$ is a pair of

- a non-empty set W, and
- an interpretation function V that assigns to every sentence letter a subset of W.

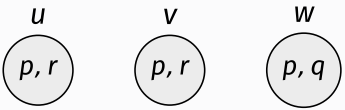
Intuitively, W is the set of worlds, and V tells us at which worlds each sentence letter is true.

Formal possible-worlds semantics

A model

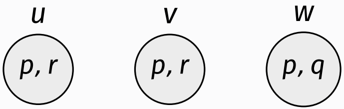
$$W = \{u, v, w\}$$

$$V(p) = \{u, v, w\}$$
$$V(q) = \{w\}$$
$$V(r) = \{u, v\}$$



Formal possible-worlds semantics

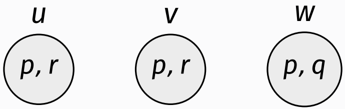
A model (M)



- Is p true at u in M?
- Is $p \wedge q$ true at u in M?
- Is $\Box q$ true at u in M?
- Is $\Diamond q$ true at u in M?
- Is $\Diamond\Diamond q$ true at u in M?

Formal possible-worlds semantics

A model (M)



- Is $p \to r$ true at w in M?
- Is $\Diamond\neg p$ true at v in M?
- Is $\Diamond\neg r$ true at v in M?
- Is $\Diamond\Box\neg(p \to r)$ true at u in M?

Formal possible-worlds semantics

Abbreviation: $M, w \models A \quad \Leftrightarrow \quad A$ is true at w in M

$$p \text{ is true at } u \text{ in } M \quad \Leftrightarrow \quad M, u \models p$$

$$p \wedge q \text{ is true at } w \text{ in } M \quad \Leftrightarrow \quad M, w \models p \wedge q$$

Formal possible-worlds semantics

Definition: Truth at a world in a model

If $M = (W, V)$ is a model, w is a member of W, ρ is any sentence letter, and A, B are any $\mathfrak{L}_M$-sentences, then

(a)	$M, w \models \rho$	iff $w \in V(\rho)$.
(b)	$M, w \models \neg A$	iff $M, w \not\models A$.
(c)	$M, w \models A \wedge B$	iff $M, w \models A$ and $M, w \models B$.
(d)	$M, w \models A \vee B$	iff $M, w \models A$ or $M, w \models B$.
(e)	$M, w \models A \rightarrow B$	iff $M, w \not\models A$ or $M, w \models B$.
(f)	$M, w \models A \leftrightarrow B$	iff $M, w \models (A \rightarrow B)$ and $M, w \models (B \rightarrow A)$.
(g)	$M, w \models \Box A$	iff $M, v \models A$ for all v in W.
(h)	$M, w \models \Diamond A$	iff $M, v \models A$ for some v in W.

Formal possible-worlds semantics

We can now give a rigorous definition of validity:

$\models A$ iff $M, w \models A$ for all (basic) models M and worlds w in M.