Logic 2: Modal Logic

Lecture 2

Wolfgang Schwarz

January 21, 2022

University of Edinburgh

Review

Review

- We've added the box $\Box$ and the diamond $\Diamond$ to the language of propositional logic.
- The box often represents some kind of necessity, the diamond some kind of possibility.
- We've talked about how to translate from English into the language of modal propositional logic.

Validity and logical validity

Validity and logical validity

What do we mean when we say that an argument is **valid**?

An argument is valid if there is no conceivable scenario in which the premises are true and the conclusion is false.

Check your understanding:

- Can an argument be valid if its conclusion is false?
- Is this argument valid? '1+1=3. Therefore: It is raining.'

Validity and logical validity

It is hot and windy.
———
It is hot.

It is hot.
———
It is not cold.

Only the first argument is **logically valid**.

An argument is logically valid if its validity does not turn on the meaning of the non-logical expressions.

- Re-interpret ‘hot’ to mean *cloudy*.
- The first argument remains valid, the second becomes invalid.

Validity and logical validity

An argument is valid if there is no conceivable scenario in which the premises are true and the conclusion false.

> An argument is logically valid if there is no conceivable scenario in which the premises are true and the conclusion is false, under any (re-)interpretation of the non-logical expressions.

Validity and logical validity

In modal logic, we treat the box and the diamond as logical expressions.

It is possible that it is raining.	$\Diamond r$
It is certain that we will get wet if it is raining.	$\Box(r \rightarrow w)$
It is possible that we will get wet.	$\Diamond w$

There is no conceivable scenario in which the premises are true and the conclusion is false, under any interpretation of the non-logical expressions.

The turnstile

The turnstile

We say that some sentences $A_1, A_2, \ldots$ **(logically) entail** a sentence A if the argument from $A_1, A_2, \ldots$ to A is logically valid.

Let's introduce an abbreviation.

$$A_1, A_2, \ldots \models B \quad \Longleftrightarrow \quad A_1, A_2, \ldots \text{ logically entail } B.$$

$\Longleftrightarrow$ There is no conceivable scenario in which $A_1, A_2, \ldots$ are all true while B is false, on any interpretation of the non-logical expressions.

The turnstile

- It is hot and windy $\models$ It is hot.
- $p \wedge q \models p$.
- $\Diamond p, \Box(p \to q) \models \Diamond q$.

The turnstile

$A_1, A_2 \ldots \models B$ iff there is no conceivable scenario and interpretation of non-logical expressions that would make all of $A_1, A_2, \ldots$ true and B false.

Informally, the turnstile says "you can't make every on the left true while making everything on the right false".

A special case: $\models B$. This says that B is (logically) valid.

A sentence is (logically) valid iff there is no conceivable scenario in which it is false, on any interpretation of the non-logical expressions.

The turnstile

- $\models p \rightarrow p$
- $\models \forall x Fx \vee \exists x \neg Fx$
- $\models \Box(p \wedge q) \rightarrow \Box p$

Systems of modal logic

Systems of modal logic

We have a language with boxes and diamonds.

Next, we need to decide which sentences in that language are valid, and which sentences are entailed by other sentences.

- Is $\Diamond p$ valid?
- Do $\Diamond r$ and $\Box(r \rightarrow w)$ entail $\Diamond w$?
- Does $\Box p$ entail p?
- ...

- $\models \Diamond p$?
- $\Diamond r, \Box(r \rightarrow w) \models \Diamond w$?
- $\Box p \models p$?
- ...

The answer depends on how we interpret the box and the diamond.

Systems of modal logic

Let's interpret the box as 'it is settled that' and the diamond as 'it is not settled that not'.

Which of these do we want in the logic of historical necessity?

- $\Box p \models p$
- $\Diamond p \models \Box p$
- $\Box p, \Box q \models \Box(p \wedge q)$
- $\Diamond p, \Diamond q \models \Diamond(p \wedge q)$
- $\Box p \models \Box\Box p$

Systems of modal logic

Suppose we want to specify for *any* sentences $A_1, A_2 \ldots, B$ of modal propositional logic whether $A_1, A_2, \ldots \models B$.

How could we do that?

Systems of modal logic

A useful fact:

$$A_1, \ldots, A_n \models B \text{ iff } \models (A_1 \wedge \ldots \wedge A_n) \rightarrow B$$

- $\Box p \models p$ iff $\models \Box p \rightarrow p$
- $\Diamond p \models \Box p$ iff $\models \Diamond p \rightarrow \Box p$
- $\Box p, \Box q \models \Box(p \wedge q)$ iff $\models (\Box p \wedge \Box q) \rightarrow \Box(p \wedge q)$
- ...

To specify which sentences entail which other sentences, it suffices to specify which sentences are valid.

Systems of modal logic

We want to specify, for every sentence of modal propositional logic, whether it is valid in the logic of historical necessity.

All instances of the following schemas are plausibly valid:

(**T**) $\Box A \rightarrow A$

(**K**) $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

(**4**) $\Box A \rightarrow \Box\Box A$

(**5**) $\Diamond A \rightarrow \Box\Diamond A$

(**Dual**) $\neg\Diamond A \leftrightarrow \Box\neg A$

Systems of modal logic

All instances of the following schemas are plausibly valid:

$$\begin{aligned}
&(\mathbf{T}) && \Box A \to A \\
&(\mathbf{K}) && \Box(A \to B) \to (\Box A \to \Box B) \\
&(\mathbf{4}) && \Box A \to \Box\Box A \\
&(\mathbf{5}) && \Diamond A \to \Box\Diamond A \\
&(\mathbf{Dual}) && \neg\Diamond A \leftrightarrow \Box\neg A
\end{aligned}$$

Also,

(**CPL**) Any truth-functional consequence of a valid sentence is valid.

(**Nec**) If A is valid then so is $\Box A$.

Example: $\Box p \models \Diamond p$.

This is equivalent to: $\models \Box p \to \Diamond p$.

1.	$\Box\neg p \to \neg p$	(T)
2.	$\neg\Diamond p \leftrightarrow \Box\neg p$	(Dual)
3.	$\neg\Diamond p \to \neg p$	(1, 2, CPL)
4.	$p \to \Diamond p$	(3, CPL)
5.	$\Box p \to p$	(T)
6.	$\Box p \to \Diamond p$	(4, 5, CPL)

This style of showing that a sentence is valid is known as an **axiomatic proof**.

A little history

A little history

In the early days of formal logic, the only known method of formal proofs was the axiomatic method.

In the axiomatic method, you lay down some axioms and inference rules.

A proof is a list of sentences each of which is either an axiom or follows from earlier sentences by one of the rules.

A little history

The Frege-Łukasiewicz axiomatization of propositional logic:

(**A1**) $A \rightarrow (B \rightarrow A)$

(**A2**) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

(**A3**) $(\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)$

(**MP**) From A and $A \rightarrow B$ one may infer B

All truth-functional tautologies are derivable from A1–A3 by MP.

There are no proofs from premises. To show that A entails B, you show that $A \rightarrow B$ is valid, by proving $A \rightarrow B$.

A little history

Axiomatic proofs are often hard to find.

1.	$p \rightarrow ((p \rightarrow p) \rightarrow p)$	(A1)
2.	$(p \rightarrow ((p \rightarrow p) \rightarrow p)) \rightarrow ((p \rightarrow (p \rightarrow p)) \rightarrow (p \rightarrow p))$	(A2)
3.	$(p \rightarrow (p \rightarrow p)) \rightarrow (p \rightarrow p)$	(1, 2, MP)
4.	$p \rightarrow (p \rightarrow p)$	(A1)
5.	$p \rightarrow p$	(3, 4, MP)

A little history

In the 1920s and 1930s, C.I. Lewis put forward a range of axiomatic "systems" of modal propositional logic, which he called S1–S5.

Each of his "systems" consisted of some axioms and rules.

Lewis's aim was to formalize reasoning about implication.
Guiding thought: A implies B iff it is impossible for A to be true and B false: $\neg\Diamond(A \land \neg B)$.
Lewis wasn't sure which principles should count as valid on the relevant understanding of 'possible'.

$$(\textbf{Dis})\quad \Diamond(A \land B) \to \Diamond A$$

$$(\textbf{4})\quad \Diamond\Diamond A \to \Diamond A$$

A little history

In 1933, Kurt Gödel gave the following axiomatization of S4:

(**PL**) The axioms of propositional logic

(**K**) $\Box(A \to B) \to (\Box A \to \Box B)$

(**T**) $\Box A \to A$

(**4**) $\Box A \to \Box\Box A$

(**MP**) From A and $A \to B$ one may infer B

(**Nec**) From A one may infer $\Box A$

If we add the axiom schema (5) $\Diamond A \to \Box\Diamond A$, we get S5.
(PL) and (MP) together are equivalent to our rule (PLC).