3 Accessibility

3.1 Variable modality

In the previous chapter, we read $\Box A$ as saying that A is true at every possible world. We might hope to allow for different flavours of modality by letting each flavour select different kinds of worlds as possible. If the box represents epistemic necessity, a possible world would be a world that is compatible with the available information. If the box represents historical necessity, a possible world would be one that can be brought about. If the box represents obligation, a possible world would be a world in which all relevant norms are respected. (These worlds are more commonly called *ideal*.)

But there is a problem. The semantics from the previous chapter determines a particular logic: S5. And that logic is not appropriate for every application of modal logic. In deontic logic, for example, we don't want the schema

(T) $\quad \Box A \to A$

to be valid. We can easily conceive of scenarios in which $\Box p$ is true (on some interpretation of p) even though p is false.

The semantics from the previous chapter renders the (T)-schema valid. Whenever a sentence $\Box A$ is true at a world w in a model then A is true at w as well, because the box quantifies over all worlds, including w. To make room for deontic logic, we need a semantics in which not all worlds in W are among the "possible" worlds over which the modal operators quantify. Not all worlds are ideal.

We might also want to allow that the worlds over which the modal operators quantify depend on the world at which the relevant sentence is evaluated. Perhaps you are obligated to do the dishes in worlds where you have promised to do the dishes, but not in worlds where you haven't made the promise. Worlds in which you don't do the dishes are then ideal relative to the second kind of world, but not relative to the first.

This kind of variability is also needed for other flavours of modality. Suppose the box quantifies over all worlds that are compatible with our knowledge. Which worlds are compatible with our knowledge depends on what we know. But we don't always know what we know. Sometimes we believe that we know something, but don't actually know it because it is false. We don't know it, without knowing that we don't know it. Among the worlds compatible with our knowledge are then worlds in which we know more than we actually do. What's compatible with our knowledge in *these* worlds is different from what's compatible with our knowledge in the actual world.

Let's assume, then, that for any world in any scenario there is a set of worlds that are possible *relative to* w. We assume that $\Box p$ as true at w iff p is true at all worlds that are possible relative to w. If a world v is possible relative to w we also say that v is **accessible** from w, or (informally) that *w can see v*.

Accessibility means different things in different applications. In epistemic logic, a world v is accessible from w iff v is compatible with what is known at w. In the logic of historical necessity, v is accessible from w iff v can be brought about at w. And so on. We can still allow for scenarios in which every worlds is accessible from every world, so that the box and the diamond are unrestricted quantifiers over all worlds in the scenario, as in the previous chapter.

Since facts about accessibility matter to the truth-value of modal sentences, they must be represented by our models. From now on, a model for $\mathfrak{L}_M$ will therefore specify which worlds in W are accessible from which others (and from themselves). This marks the difference between a "basic model" and a "Kripke model" – named after Saul Kripke, who popularised models of this kind.

Definition 3.1

A **Kripke model** of $\mathfrak{L}_M$ is a triple $\langle W, R, V \rangle$ consisting of

- a non-empty set W,
- a binary relation R on W, and
- a function V that assigns to each sentence letter of $\mathfrak{L}_M$ a subset of W.

R is the accessibility relation. It is called a relation "on W" because it holds between members of W. We write 'wRv' to express that R holds between w and v.

We also need to update definition 2.2, which settles under what conditions an $\mathfrak{L}_M$-sentence is true at a world in a model. The old definition had the following clauses for the box and the diamond:

(g) $M, w \models \Box A$ iff $M, v \models A$ for all v in W.
(h) $M, w \models \Diamond A$ iff $M, v \models A$ for some v in W.

In the new semantics, the box and the diamond only quantify over accessible worlds:

(g) $M, w \models \Box A$ iff $M, v \models A$ for all v in W such that wRv.
(h) $M, w \models \Diamond A$ iff $M, v \models A$ for some v in W such that wRv.

Here is the full definition, for completeness.

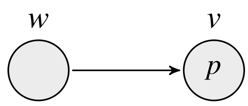
Definition 3.2: Kripke Semantics

If $M = \langle W, R, V \rangle$ is a Kripke model, w is a member of W, P is any sentence letter, and A, B are any $\mathfrak{L}_M$-sentences, then

(a) $M, w \models P$ iff w is in $V(P)$.
(b) $M, w \models \neg A$ iff $M, w \not\models A$.
(c) $M, w \models A \land B$ iff $M, w \models A$ and $M, w \models B$.
(d) $M, w \models A \lor B$ iff $M, w \models A$ or $M, w \models B$.
(e) $M, w \models A \to B$ iff $M, w \not\models A$ or $M, w \models B$.
(f) $M, w \models A \leftrightarrow B$ iff $M, w \models A \to B$ and $M, w \models B \to A$.
(g) $M, w \models \Box A$ iff $M, v \models A$ for all v in W such that wRv.
(h) $M, w \models \Diamond A$ iff $M, v \models A$ for some v in W such that wRv.

When I speak of truth at a world in a Kripke model, this should always be understood in accordance with definition 3.2. Definition 2.2 defines truth at a world in a basic model.

To see definition 3.2 in action, consider a simple model with two worlds, w and v. World v is accessible from world w, but v is not accessible from w. Neither world can access itself. The interpretation function assigns $\{w\}$ to p and the empty set $\emptyset$ to all other sentence letters. The model can be pictured as follows, with an arrow representing accessibility:



Using definition 3.2, we can figure which $\mathfrak{L}_M$-sentences are true at which worlds in the model. For example:

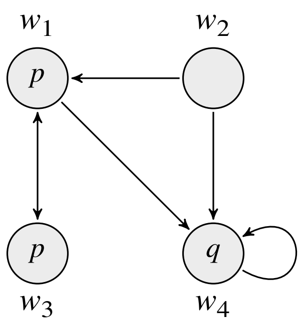
- By clause (a) of definition 3.2, p is true at v and false at w.
- By clause (h), $\Diamond p$ is true at w because p is true at v and v is accessible from w. $\Diamond p$ is false at v because there is no world accessible from v at which p is true.
- By clause (g), $\Box\Diamond p$ is false at w because $\Diamond p$ is false at v and v is accessible from w. $\Box\Diamond p$ is true at v because there is no world accessible from v at which $\Diamond p$ is false.

Note that $\Diamond p$ and $\Box\Diamond p$ have different truth-values at w (and at v). In the new semantics, we can no longer ignore all but the last in a string of modal operators. Note also that $\Box p$ is true at w even though p is false; $\Box p \to p$ is no longer valid.

Exercise 3.1

Explain why every sentence of the form $\Box A$ is true at world v in the above model.

The next three exercises refer to the following model:



Exercise 3.2

At which worlds in the model are the following sentences true?

(a) $p \lor \neg q$

(b) $\Box(p \lor \neg q)$

(c) $\Diamond(\neg p \wedge \neg q)$
(d) $\Diamond\Box q$
(e) $\Diamond\Diamond\Box q$

Exercise 3.3

For each world in the model, find an $\mathfrak{L}_M$-sentence that is true only at that world.

Exercise 3.4

Can you draw a diagram of a smaller model (with fewer worlds) in which the exact same $\mathfrak{L}_M$-sentences are true at w_1?

3.2 The systems K and S5

As in the previous chapter, we call a sentence *valid* if it is true at all worlds in all models. But we now use a different conception of models, and a different definition of truth at a world in a model. To avoid confusion, it is best to use different expressions for different kinds of validity. Let's call the new kind of validity *K-validity*. ('K' for Kripke.) The old kind will henceforth be called *S5-validity*, because the sentences that are valid by the definition from the previous chapter are precisely the sentences in C.I. Lewis's system S5.

Definition 3.3

A sentence A is **K-valid** (for short, $\models_K A$) iff A is true at every world in every Kripke model.

The same distinction applies to the concept of entailment. Entailment in the old sense (definition 2.4) will henceforth be called *S5-entailment*. Our new definition of models and truth lead to the concept of *K-entailment*.

Definition 3.4

Some sentences Γ **K-entail** a sentence A (for short: $\Gamma \models_K A$) iff there is no world in any Kripke model at which all sentences in Γ are true while A is false.

The set of K-valid sentences is a system of modal logic. This system did not figure in C.I. Lewis's list of systems. It is known as **system K**.

K is **weaker** than S5, by which we mean that not all S5-valid sentences are K-valid. $\Box p \to p$, for example, is S5-valid but not K-valid. Conversely, however, every K-valid sentence is S5-valid. Let's prove this.

Observation 3.1: Every K-valid sentence is S5-valid.

Proof: In essence, observation 3.1 holds because the basic models from the previous chapter can be simulated by Kripke models in which all worlds have access to all worlds. If a sentence A is K-valid, meaning that A is true throughout every Kripke model, then A is true throughout every Kripke model of this kind, and so A is also true in every basic model.

It is worth going through this more carefully. For any basic model $M = \langle W, V \rangle$, let M^* be the Kripke model $\langle W, R, V \rangle$ with the same worlds W and the same interpretation function V, and with an accessibility relation R that holds between all worlds in W. That is, every world in M^* can see every other world as well as itself. If every world can see every world, then it makes no difference whether we use definition 2.2 or definition 3.2 to evaluate the truth of sentences at a world. That's because the two definitions only differ for the case of the modal operators, which definition 2.2 interprets as quantifiers over all worlds, while definition 3.2 interprets them as quantifiers over the accessible worlds. So we have:

(*) A sentence is true at a world w in a basic model M iff it is true at w in the corresponding Kripke model M^*.

(A full proof of (*) would proceed by induction on complexity of the sentence.)

Now suppose a sentence A is *not* S5-valid, meaning that it is false at some world w in some basic model M. By (*), it follows that A is also false at some world in some Kripke model – namely, at the same world w in M^*. And if A is false at some world in some Kripke model, then A is not K-valid. By contraposition, it follows that if A is K-valid, then A is S5-valid. □

You may remember from section 1.5 that S5 can be axiomatized by five axiom schemas and two rules:

(Dual)	$\neg\Diamond A \leftrightarrow \Box\neg A$
(K)	$\Box(A \to B) \to (\Box A \to \Box B)$
(T)	$\Box A \to A$
(4)	$\Box A \to \Box\Box A$
(5)	$\Diamond A \to \Box\Diamond A$
(Nec)	If A is in the system, then so is $\Box A$.
(CPL)	If $\Gamma \models_P A$ and all members of Γ are in the system, then so is A.

All instances of (Dual), (K), (T), (4), and (5) are S5-valid, and all and only the S5-valid sentences can be derived from instances of these axioms by (Nec) and (CPL).

The system K can be axiomatized by dropping three of the axiom schemas: (T), (4), and (5), leaving only (Dual) and (K). All and only the K-valid sentences can be derived from instances of (Dual) and (K) by (Nec) and (CPL).

(Many authors define $\Box$ as $\neg\Diamond\neg$ or $\Diamond$ as $\neg\Box\neg$, in which case (Dual) is true by definition. The only remaining axiom schema is then (K). Don't confuse the schema (K) with the system K!)

Exercise 3.5

(a) Describe a Kripke model in which some instance of (4) is false at some world.

(b) Describe a Kripke model in which some instance of (5) is false at some world.

Exercise 3.6

Can you find an instance of the (T)-schema that is K-valid?

Exercise 3.7

Show that $\Box(p \lor \neg p)$ is K-valid, using definition 3.2.

3.3 Some other normal systems

For many applications of modal logic, we need a concept of validity that lies in between K-validity and S5-validity. Suppose, for example, we read the box as physical necessity and the diamond as physical possibility, understood as compatibility with the laws of nature. On a popular conception of what it means to be a law of nature, nothing that happens is ever incompatible with the laws of nature. Equivalently, anything that is physically necessary is actually the case. We therefore want $\Box A$ to entail A. On the other hand, it is not clear if $\Box A$ should entail $\Box\Box A$: if A is physically necessary, can we infer that it is physically necessary that A is physically necessary? Below I will argue that we can't. If that is right, then the logic of physical necessity is neither K nor S5. We want a logic with (T) ($\Box A \to A$) but without (4) ($\Box A \to \Box\Box A$). S5 gives us both, K gives us neither.

Our current semantics makes it easy to define systems in between K and S5 by putting restrictions on the accessibility relation in Kripke models.

Let's say that an $\mathfrak{L}_M$-sentence is **valid in a class of Kripke models** iff the sentence is true at every world in every model that belongs to the class. K-validity is validity in the class of all Kripke models. S5-validity is validity in the class of Kripke models in which every world has access to every world (as mentioned earlier, in the proof of observation 3.1).

If you inspect countermodels to the K-validity of $\Box p \to p$, you may notice that all of them involve worlds that don't have access to themselves. If we require that every world can see itself then all instances of the (T)-schema become valid.

Observation 3.2: All instances of (T) are valid in the class of Kripke models in which every world is accessible from itself.

Proof: According to clause (e) of definition 3.2, an instance of $\Box A \to A$ is false at a world w only if $\Box A$ is true at w and A is false; but if $\Box A$ is true at w and w has access to itself, then by clause (g) of definition 3.2, A is true at w. So if $\Box A \to A$ is false at w, and w is accessible from itself, then A is both true and false at w, which is impossible. Hence $\Box A \to A$ is true at every world in every model in which every world is accessible from itself. □

A relation R on a set W is called **reflexive** if each member of W is R-related to itself. If the accessibility relation in a Kripke model is reflexive, we'll also call the model itself reflexive. Observation 3.2 therefore states that all instances of (T) are valid in the class of reflexive Kripke models.

The set of all sentences that are valid in the class of reflexive Kripke models is known as **system T**. Accordingly, any sentence that is valid in this class of Kripke models (every member of system T) is called **T-valid**.

System T is stronger than K, but weaker than S5. The system can be axiomatized by adding the axiom schema (T) to the axioms and rules of K. We don't have (4) or (5). $\Box p \to \Box\Box p$ is S5-valid but not T-valid.

Systems of modal logic sometimes share their name with a schema. For disambiguation, I always put schema names in parentheses. (T) is a schema, T is a system. (K) is a schema, K is a system. All instances of (T) are in T, but many sentences in T – for example, all instances of (K) – are not instances of (T).

Exercise 3.8

Show that $\Box p \to \Diamond p$ is T-valid.

In chapter 7, we will study a temporal application of modal logic in which the box is read as 'it is always going to be the case that'. The "worlds" in a Kripke model here represent times. $\Box p$ is understood to be true at a time t iff p is true at all times after t. The accessibility relation is the earlier-later relation: $t_1 R t_2$ iff t_1 is earlier than t_2. In this application, we don't want to assume that R is reflexive, which would mean that every point in time is earlier than itself. But we'll want something else. Suppose t_1 is earlier than t_2, and t_2 is earlier than t_3. Then surely t_1 is earlier than t_3.

A relation R is called **transitive** if whenever xRy and yRz then xRz. As before, we call a Kripke model transitive if its accessibility relation is transitive. When we do temporal logic, we will restrict the relevant models to transitive models.

The set of sentences that are valid in the class of transitive Kripke models is known as **system K4**. The name alludes to the fact that this system can be axiomatized by adding schema (4) to the axioms and rules of K.

Observation 3.3: All instances of (4) are valid in the class of transitive Kripke models.

Proof: Suppose for reductio that there is some transitive Kripke model in which some instance of $\Box A \to \Box\Box A$ is false at some world w. By clause (e) of definition 3.2, it follows that (i) $\Box A$ is true at w and (ii) $\Box\Box A$ is false at w. By clause (g) of definition 3.2, (ii) implies that there is some world v accessible from w where $\Box A$ is false. And that, in turn implies that there is some world u accessible from v at which A is false. Since R is transitive, u is accessible from w. By (i), A is true at u. So A is both true and false at u. Contradiction. □

We can combine the systems T and K4 by requiring both reflexivity and transitivity. The set of sentences valid in the class of reflexive and transitive Kripke models is C.I. Lewis's **system S4**. It is stronger than K, T, and K4, but weaker than S5.

There are many other conditions we could impose on the accessibility relation, and many combinations of these conditions. Each of them defines a system of modal logic. The following table lists some well-known model classes with the conventional names for the corresponding systems, repeating (for future reference) the ones we already know. We will have a closer look at some of these systems in later chapters, when we turn to applications of modal logic.

System	*Constraint on R*
K	–
T	R is **reflexive**: every world in W can access itself
D	R is **serial**: every world in W can access some world
K4	R is **transitive**: whenever wRv and vRu, then wRu
K5	R is **euclidean**: whenever wRv and wRu, then vRu
KD45	R is serial, transitive, and euclidean
B	R is reflexive and **symmetric**: whenever wRv then vRw
S4	R is reflexive and transitive
S4.2	R is reflexive, transitive, and **convergent**: whenever wRv and wRu, then there is some t such that vRt and uRt
S5	R is reflexive, transitive, and symmetric
S5	R is **universal**: every world has access to every world

S5 occurs twice in the list. We already know S5 as the system for universal models, in which the box and the diamond quantify unrestrictedly over the whole space W. But we also get S5 if we merely require the accessibility relation to be reflexive, transitive, and symmetric.

Relations that are reflexive, transitive, and symmetric are called **equivalence relations**. An equivalence relation on a set divides the members of the set into classes within which everything stands in the relation to everything. (These classes are called **equivalence classes**.)

For example, let S be the relation that holds between two people iff they have the same birthday. This is an equivalence relation. It is reflexive: everyone has the same birthday as themselves. It is transitive: if aSb and bSc then aSc. And it is symmetric: if aSb then bSa. For any person a, consider the class $[a]_S$ of everyone who has the same birthday as a. (A "class" is essentially the same thing as a set.) Everyone in $[a]_S$ has the same birthday as everyone else in $[a]_S$. So within $[a]_S$, the same-birthday relation S is universal.

Now let me explain why the above two characterisations of S5 are equivalent.

Observation 3.4: A sentence is valid in the class of Kripke models whose accessibility relation is universal iff it is valid in the class of Kripke models whose accessibility relation is an equivalence relation.

Proof sketch: The right-to-left direction is easy. If R is the universal relation on W, then R is reflexive, transitive, and symmetric. So the universal relation on W is a special kind of equivalence relation on W. If a sentence is valid in every model in which R is an equivalence relation, it must therefore be valid in every model in which R is universal.

The other direction is more interesting. We argue by contraposition, showing that if a sentence A is not valid in the class of models in which R is an equivalence relation, then R is also not valid in the class of universal models. So assume A is not valid in the class of models in which R is an equivalence relation. Then there is some world w in some such model $M = \langle W, R, V \rangle$ such that $M, w \not\models A$. Define a new model $M' = \langle W', R', V' \rangle$ as follows:

> W' is the class of worlds accessible in M from w (i.e., the equivalence class $[w]_R$).
>
> R' is the universal relation on W'.
>
> V' is the restriction of V to W', so that for any sentence letter B, $V'(B) = V'(B) \cap W'$.

(If X and Y are sets, then $X \cap Y$ – the *intersection* of X and Y – is the set of all things that are both in X and in Y.)

M' has a universal accessibility relation. But from the perspective of w, M and M' are indistinguishable. *Any sentence is true at w in M iff it is true at w in M'.* This could be shown by induction, but I hope you see intuitively why it is the case.

Granting the italicized sentence, the assumption that A is false at some model whose accessibility relation is an equivalence relation entails that A is false in some model whose accessibility relation is universal. □

Exercise 3.9

Let R be the relation on the set of people that holds between a and b iff b is at least as old as a. Is R reflexive? serial? transitive? euclidean? symmetric? universal?

Exercise 3.10

Explain these facts:

(a) If R is symmetric and transitive, then R is euclidean.
(b) If R is symmetric and euclidean, then R is transitive.
(c) If R is reflexive and euclidean, then R is symmetric.

Exercise 3.11

What is wrong with the following argument? "If R is symmetric, then wRv implies vRw; if R is transitive, it follows that wRw. So symmetry and transitivity together imply reflexivity."

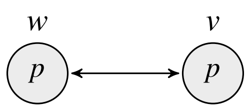
3.4 Frames

There is a close connection between conditions on the accessibility relation in Kripke models and modal schemas – between reflexivity and the (T)-schema, between transitivity and the (4)-schema, and so on. What exactly is that connection?

You might think the connection between (T) and reflexivity is this:

(?) All instances of (T) are valid in a models iff the model is reflexive.

But that's false. We know (observation 3.2) that all (T) instances are valid in the class of reflexive models. It follows that all (T) instances are valid in every reflexive model. But the other direction fails. There are non-reflexive models in which all (T) instances are valid. The following model is an example.



There are two worlds, both of which can see each other; neither can see itself. p is true at both worlds, all other sentence letters are false at both worlds. This model is not reflexive, but no instance of the (T)-schema $\Box A \to A$ is false at any world in the model. (Try to find a false instance!) The fact that the (T)-schema is valid in a class of models therefore does not entail that all models in the class are reflexive. The class might contain models like the one just described.

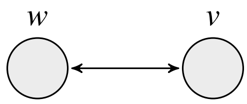
To understand the connection between modal schemas and conditions on the accessibility relation, we need to talk about *frames*. A frame is what you get if take a model and remove the interpretation function.

Definition 3.5

A **Kripke frame** is a pair of a non-empty set W and a relation R on W.

Roughly speaking, if we think of a model as representing a scenario and an interpretation, then a frame is the part of the model that represents the scenario.

Frames can be pictured just like Kripke models, but without any sentence letters in the nodes. The frame of the model displayed above looks like this:



Now remember that validity is truth in virtue of the meaning of the logical expressions. Whether a sentence is valid should not depend on the meaning of the non-logical expressions. So if we define a particular kind of validity by reference to a class of Kripke models, the constraints we impose on the models in the class should be constraints on the frame of the models, not on the interpretation function.

To see why, suppose I suggested that a sentence is "X-valid" iff it is true at all worlds in all Kripke model whose interpretation function assigns the empty set to the sentence letter p. So $\Box\neg p$ is X-valid, while $\Box\neg q$ is X-invalid. But $\Box\neg p$ and $\Box\neg q$ have the same logical form. If $\Box\neg p$ is true in virtue of its logical form, then $\Box\neg q$ should also be true in virtue of its logical form. X-validity is not a sensible concept of logical validity. The systems from the previous section were all defined sensibly, by putting constraints on the frame of a Kripke model, not on the interpretation function.

Let's say that a sentence is **valid on a frame** if it is true at all worlds in all models with that frame. A sentence is **valid in a class of frames** if it valid on all frames in the class.

If a sentence is valid in the class of all models whose accessibility relation satisfies a certain condition, then it is also valid in the class of all frames whose accessibility relation satisfies that condition, and vice versa. We could have defined the systems from the previous section in terms of frame classes rather than model classes: K is the set of sentences valid in the class of all frames, T is the set of sentences valid in the class of reflexive frames, and so on. (A reflexive/transitive/etc. frame is a frame with a reflexive/transitive/etc. accessibility relation.)

Now here is the connection between (T) and reflexivity: All (T) instances are valid in a class of frames iff every frame in the class is reflexive. More simply:

Observation 3.5: All instances of (T) are valid on a frame iff the frame is reflexive.

Proof: The right-to-left direction follows from observation 3.2, according to which all (T) instances are valid in the class of reflexive models, and therefore in the class of reflexive frames, and therefore on any frame in that class. For the other direction, we have to show that if all instances of (T) are valid on a frame $\langle W, R \rangle$, then R is reflexive. We do this by showing that if R is not reflexive, then we can find an interpretation function V that makes $\Box p \to p$ false at some world w. w will be an arbitrary world in W that can't see itself. (There must be some such world if R is not reflexive.) Let $V(p)$ comprise all worlds in W except w. Then $\Box p$ is true at w and p false. So $\Box p \to p$ is false at w. □

If all instances of a schema are valid on all and only the frames whose accessibility relation satisfies a certain property, the schema is said to **correspond** to that property (and to *define* the relevant class of frames). Observation 3.5 says that the (T) schema corresponds to reflexivity.

Instead of proving more facts about the correspondence between modal schemas and frame conditions, I will simply give you a list of some important results.

Schema		*Corresponding Frame Condition*
(T)	$\Box A \to A$	R is reflexive: every world in W is accessible from itself
(D)	$\Box A \to \Diamond A$	R is serial: every world in W can access some world in W
(B)	$A \to \Box\Diamond A$	R is symmetric: whenever wRv then vRw
(4)	$\Box A \to \Box\Box A$	R is transitive: whenever wRv and vRu, then wRu
(5)	$\Diamond A \to \Box\Diamond A$	R is euclidean: whenever wRv and wRu, then vRu
(G)	$\Diamond\Box A \to \Box\Diamond A$	R is convergent: whenever wRv and wRu, then there is some t such that vRt and uRt

Correspondence facts are often useful when trying to figure out which schemas should be valid on a given interpretation of the modal operators. Return to the case of physical possibility and necessity from the start of section 3.3. I claimed that on this interpretation of the box and the diamond, we should not regard all instances of the (4)-schema $\Box A \to \Box\Box A$ as valid. My claim is not based on a direct intuition that something could be physically necessary without it being physically necessary that it is physically necessary. My claim is rather based on a judgement about the non-transitivity of physical accessibility. My reasoning goes like this. I assume that a world v is physically possible relative to a world w if nothing that happens at v contradicts the laws of nature at w. This does not imply that v has the same laws as w. For example, suppose the only law at w is that ravens are black; at v, there is no such law but there happen to be no non-black ravens. Then what happens at v does not contradict the laws at w, even though v has different laws. Relative to the laws of v, worlds with white ravens are physically possible. So a world accessible from a world that is accessible from w need not itself be accessible from w. Since (4) corresponds to transitivity, I can infer that the logic of physical necessity does not render all instances of that schema valid.

Exercise 3.12

Can you find frame conditions that correspond to these schemas?

(a) $\Box A \leftrightarrow A$

(b) $\Box A$

3.5 More trees

In section 2.5, I described the tree method for checking whether a sentence is valid, and for constructing countermodels. These were the rules for the box and the diamond:

$\Box A \quad (\omega)$	$\Diamond A \quad (\omega)$	$\neg\Box A \quad (\omega)$	$\neg\Diamond A \quad (\omega)$
$A \quad (\nu)$	$A \quad (\nu)$	$\neg A \quad (\nu)$	$\neg A \quad (\nu)$
↑ old	↑ new	↑ new	↑ old

The rule for $\Box A$ allows us to infer, from the hypothesis that $\Box A$ is true at some world, that A is true at any world that occurs on a tree branch. This made sense given the semantics of the previous chapter, where the box quantified unrestrictedly over all worlds. With the new semantics of the present chapter, we need to change the tree rules.

If $\Box A$ is true at a world w, and there's some other world v on the branch, we can only infer that A is true at v if v is accessible from w. So we need to keep track of which worlds are accessible from any world on a tree. We do this by adding meta-linguistic statements about accessibility to the tree.

For example, suppose we want to expand the following node.

n. $\quad \Diamond p \quad (w)$

The node represents the hypothesis that $\Diamond p$ is true at w. It follows that p is true at some world v. Moreover, that world v must be accessible from w. So we add two new nodes:

m. $\quad wRv$

m+1. $\quad p \quad (v)$

Node $m+1$ is what we would have added by the old rules. Node m is a meta-linguistic statement reminding us that v is accessible from w. 'wRv' is not a sentence of $\mathfrak{L}_M$; it isn't true or false relative to a world, which is why node m has no world label.

What if we want to expand a box node?

n. $\Box p$ (w)

By itself, this doesn't tell us anything about the truth-value of p at any world. We can't infer that p is true at w, because w might not be accessible from itself. Indeed, if no world is accessible from w, then $\Box p$ can be true even if p is false at every world. So we can't even infer that there is some world or other at which p is true.

However, suppose a branch that contains node n also contains the following node.

m. wRv

Now we can infer that p is true at v. So to expand a box node on a branch, there must be another node on the branch telling us that the world w at which the boxed sentence is true has access to some world v.

Here are diagrams of the new rules for the box and the diamond.

$\Box A$ (ω)	$\Diamond A$ (ω)	$\neg\Box A$ (ω)	$\neg\Diamond A$ (ω)
$\omega R \nu$			$\omega R \nu$
	$\omega R \nu$	$\omega R \nu$	
A (ν)	A (ν)	$\neg A$ (ν)	$\neg A$ (ν)
	↑ new	↑ new	

If two nodes occur above the dotted line in a rule, as in the rule for $\Box A$, this means that the rule can only be applied if both nodes already occur on the relevant branch (in any order, and not necessarily adjacent to each other).

The rules for negated boxes and diamonds are what you would expect from the duality of the box and the diamond. Note that only nodes of type $\Diamond A$ and $\neg\Box A$ allow us to introduce hypotheses about accessibility into a tree.

The rule for the classical connectives all stay the same. Together, all these rules are known as the **K-rules**; the tree rules from section 2.5 are the **S5-rules**.

Here is a schematic tree proof to show that $\models_K \Box(A \wedge B) \to (\Box A \wedge \Box B)$.

1.	$\neg(\Box(A \land B) \to (\Box A \land \Box B))$	(w)	(Ass.)
2.	$\Box(A \land B)$	(w)	(1)
3.	$\neg(\Box A \land \Box B)$	(w)	(1)

Left branch:

4.	$\neg\Box A$	(w)	(3)
6.	wRv		(4)
7.	$\neg A$	(v)	(4)
8.	$A \land B$	(v)	(2,6)
9.	A	(v)	(8)
10.	B	(v)	(8)
	×		

Right branch:

5.	$\neg\Box B$	(w)	(3)
11.	wRu		(5)
12.	$\neg B$	(u)	(5)
13.	$A \land B$	(u)	(2,11)
14.	A	(u)	(13)
15.	B	(u)	(13)
	×		

The annotation '(2,6)' for node 8 indicates that this node is based on two assumptions from earlier in the branch: the assumption on node 2 that $\Box(A \land B)$ is true at w, and the assumption on node 6 that wRv. Only these two assumptions together allow us to infer that $A \land B$ is true at v.

What happens if we try to prove $\Box p \to p$?

1.	$\neg(\Box p \to p)$	(w)	(Ass.)
2.	$\Box p$	(w)	(1)
3.	$\neg p$	(w)	(1)

At this point, no more rules can be applied. We can read off a countermodel from the open branch:

$$W = \{w\}$$
$$R = \emptyset$$
$$V(p) = \emptyset$$

This is the smallest possible Kripke model. It consists of a single world that can't see itself. '$R = \emptyset$' is a way of saying that no world can see any world. If you want to say that R holds between w and v and between v and u, you might write '$R = \{(w, v), (v, u)\}$' or simply 'wRv, vRu'.

Exercise 3.13

Use the K-rules to check which of the following sentences are K-valid. If a sentence is invalid, describe a countermodel.

(a) $(\Box p \land \Box q) \to \Box(p \land q)$
(b) $\Diamond(p \land q) \to (\Diamond p \land \Diamond q)$
(c) $(\Diamond p \land \Diamond q) \to \Diamond(p \land q)$
(d) $\Diamond(p \lor q) \leftrightarrow (\Diamond p \lor \Diamond q)$
(e) $\Box(p \lor q) \leftrightarrow (\Box p \lor \Box q)$
(f) $\Box(p \to q) \to (\Diamond p \to \Diamond q)$.
(g) $(\Box p \land \Diamond q) \to \Diamond(p \land q)$.

For systems in between K and S5 that are characterised by certain constraints on the accessibility relation, we add new rules for manipulating accessibility nodes. For example, if we want to check whether a sentence is T-valid, we use a *reflexivity rule* in addition to the K-rules. The reflexivity rule says that if a world variable ω occurs on a branch, then we may always add $\omega R \omega$ to the branch.

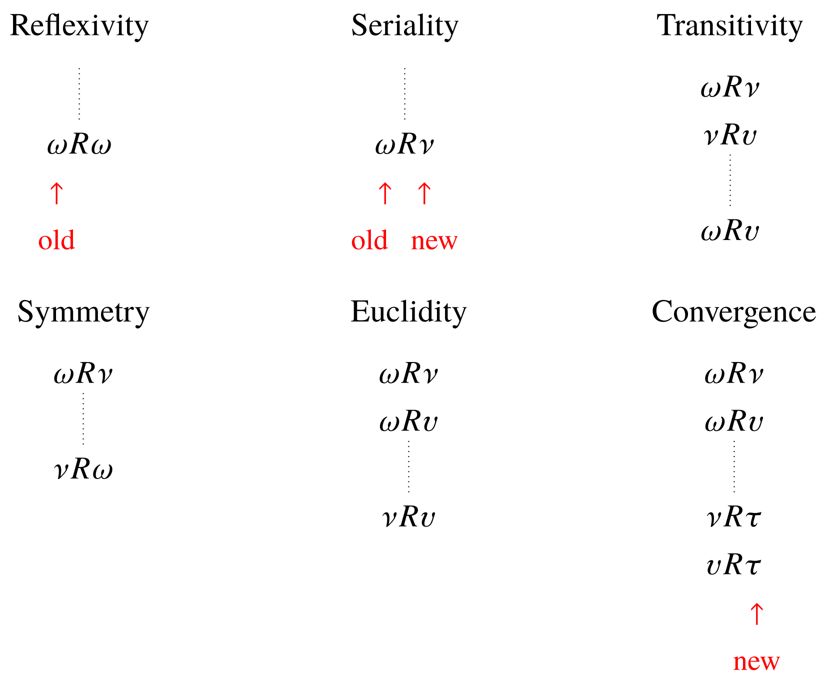
Here is a proof of $\Box p \to p$, using the reflexivity rule.

1.	$\neg(\Box p \to p)$	(w)	(Ass.)
2.	$\Box p$	(w)	(1)
3.	$\neg p$	(w)	(1)
4.	wRw		(Ref.)
5.	p	(w)	(2,4)
	x		

To test for validity in the class of transitive frames (or models), we need a *transitivity rule*, which allows us to infer $\omega R \upsilon$ from $\omega R \nu$ and $\nu R \upsilon$. Here is a proof of $\Box p \to \Box\Box p$ that uses this rule.

1.	$\neg(\Box p \to \Box\Box p)$	(w)	(Ass.)
2.	$\Box p$	(w)	(1)
3.	$\neg\Box\Box p$	(w)	(1)
4.	wRv		(3)
5.	$\neg\Box p$	(v)	(3)
6.	vRu		(5)
7.	$\neg p$	(u)	(5)
8.	wRu		(4,6,Tr.)
9.	p	(u)	(2,8)
	×		

The following diagrams summarize the tree rules for the frame conditions we have so far considered.



By selectively adding some of these rules to the K-rules, we get tree rules for a variety of modal logics. (Compare the table on p. 61.)

System	*Tree Rules*
K	K-rules
T	K-rules and reflexivity rule
D	K-rules and seriality rule
K4	K-rules and transitivity rule
K5	K-rules and euclidity rule
KD45	K-rules, seriality rule, transitivity rule, and euclidity rule
B	K-rules, reflexivity rule, and symmetry rule
S4	K-rules, reflexivity rule, and transitivity rule
S4.2	K-rules, reflexivity rule, transitivity rule, and convergence rule

Exercise 3.14

Use the tree method to check the following claims.

(a) $\models_{K4} \Diamond p \rightarrow \Diamond\Diamond p$.
(b) $\models_{D} (\Box p \wedge \Box q) \rightarrow \Diamond(p \vee q)$.
(c) $\models_{B} \Diamond p \rightarrow \Box\Diamond p$.
(d) $\models_{T} (\Diamond\Box(p \rightarrow q) \wedge \Box p) \rightarrow \Diamond q$.
(e) $\models_{T} \Diamond(p \rightarrow \Box\Diamond p)$.