Chapter 7

Exercise 7.1

(a) $\mathsf{H}\,\neg p$
p: It is warm

(b) $\mathsf{F}\,p$
p: There is a sea battle

(c) $\neg\,\mathsf{F}\,\mathsf{P}\,p$ or, perhaps, $\mathsf{F}\,\neg\,\mathsf{P}\,p$
p: There is a sea battle

(d) $\mathsf{F}(p \vee \mathsf{P}\,q)$ or $\mathsf{F}(\mathsf{F}\,p \vee \mathsf{F}\,\mathsf{P}\,q)$
p: It is warm

(e) $\neg\,\mathsf{P}\,p \to \neg\,\mathsf{F}\,q$ or $\mathsf{G}(\neg\,\mathsf{P}\,p \to \neg q)$
p: You study, q: you pass the exam

(f) $\mathsf{P}(p \wedge q)$
p: I am having tea, q: the door bell rings

Exercise 7.2

(a), (c), (f), (g), and (h) are true, (b), (d), and (e) are false.

Exercise 7.3

(Con1): Suppose some sentence of the form $A \to \mathsf{G}\,\mathsf{P}\,A$ is false at some time t in some temporal model. By clause (e) of definition 7.2, this means that A is true at t and $\mathsf{G}\,\mathsf{P}\,A$ is false at t. By clause (h), the latter means that there is a time s with $t < s$ such that $\mathsf{P}\,A$ is not true at s. By clause (i), it follows that A is not true at t. Contradiction.

The argument for (Con2) is analogous.

Exercise 7.4

(a)

1.	$\neg(A \to \mathsf{G}\,\mathsf{P}A)$	(t)	(Ass.)
2.	A	(t)	(1)
3.	$\neg\,\mathsf{G}\,\mathsf{P}A$	(t)	(1)
4.	$t < s$		(3)
5.	$\neg\,\mathsf{P}A$	(s)	(3)
6.	$\neg A$	(t)	(4,5)
	×		

(b)

1.	$\neg(A \to \mathsf{H}\,\mathsf{F}A)$	(t)	(Ass.)
2.	A	(t)	(1)
3.	$\neg\,\mathsf{H}\,\mathsf{F}A$	(t)	(1)
4.	$s < t$		(3)
5.	$\neg\,\mathsf{F}A$	(s)	(3)
6.	$\neg A$	(t)	(4,5)
	×		

(c)

1.	$\neg(\mathsf{F}A \to \mathsf{H}\,\mathsf{F}\,\mathsf{F}A)$	(t)	(Ass.)
2.	$\mathsf{F}A$	(t)	(1)
3.	$\neg\,\mathsf{H}\,\mathsf{F}\,\mathsf{F}A$	(t)	(1)
4.	$s < t$		(3)
5.	$\neg\,\mathsf{F}\,\mathsf{F}A$	(s)	(3)
6.	$\neg\,\mathsf{F}A$	(t)	(4,5)
	×		

(d)

1.	$\neg(\mathsf{P}\,\mathsf{G}A \to \mathsf{P}\,\mathsf{F}A)$	(t)	(Ass.)
2.	$\mathsf{P}\,\mathsf{G}A$	(t)	(1)
3.	$\neg\,\mathsf{P}\,\mathsf{F}A$	(t)	(1)
4.	$s < t$		(2)
5.	$\mathsf{G}A$	(s)	(2)
6.	A	(t)	(4,5)
7.	$\neg\,\mathsf{F}A$	(s)	(3,4)
8.	$\neg A$	(t)	(4,7)
	×		

(e)

1. $\neg(\mathsf{H}A \leftrightarrow \mathsf{H}\,\mathsf{F}\,\mathsf{H}A)$ (t) (Ass.)

2.	$\mathsf{H}A$	(t)	(1)	4.	$\neg\,\mathsf{H}A$	(t)	(1)
3.	$\neg\,\mathsf{H}\,\mathsf{F}\,\mathsf{H}A$	(t)	(1)	5.	$\mathsf{H}\,\mathsf{F}\,\mathsf{H}A$	(t)	(1)
6.	$s < t$		(3)	9.	$s < t$		(4)
7.	$\neg\,\mathsf{F}\,\mathsf{H}A$	(s)	(3)	15.	$\neg A$	(s)	(4)
8.	$\neg\,\mathsf{H}A$	(t)	(6,7)	11.	$\mathsf{F}\,\mathsf{H}A$	(s)	(5,9)
	×			12.	$s < r$		(11)
				13.	$\mathsf{H}A$	(r)	(11)
				14.	A	(s)	(12,13)
					×		

Exercise 7.5

Suppose $<$ is transitive, and $x > y$ and $y > z$. Equivalently, $y < x$ and $z < y$.By transitivity of $<$, we have $z < x$. So $x > z$.

Exercise 7.6

Suppose R is transitive. If there are points x and y for which xRy and yRx then xRx by transitivity. So if R isn't asymmetric then it isn't irreflexive. If R isn't irreflexive then there is a point x with xRx. This violates asymmetry, because asymmetry demands that if xRx then not xRx.

Exercise 7.7

If time is transitive and circular, then it is neither asymmetric nor irreflexive.

Exercise 7.8

(a), (d), and (e) are invalid. Here are trees for (b), (c), and (f):

(b)	1.	$\neg(\mathsf{P}\,\mathsf{G}\,\mathsf{G}\,p \to \mathsf{G}\,\mathsf{G}\,p)$	(t)	(Ass.)
	2.	$\mathsf{P}\,\mathsf{G}\,\mathsf{G}\,p$	(t)	(1)
	3.	$\neg\,\mathsf{G}\,\mathsf{G}\,p$	(t)	(1)
	4.	$s < t$		(2)
	5.	$\mathsf{G}\,\mathsf{G}\,p$	(s)	(2)
	6.	$t < r$		(3)
	7.	$\neg\,\mathsf{G}\,p$	(r)	(3)
	8.	$s < r$		(3,6)
	9.	$\mathsf{G}\,p$	(r)	(5,8)
		×		

(c)

1.	$\neg(\mathsf{P}\,\mathsf{F}\,p \to (\mathsf{P}\,p \vee (p \vee \mathsf{F}\,p)))$	(t)	(Ass.)
2.	$\mathsf{P}\,\mathsf{F}\,p$	(t)	(1)
3.	$\neg(\mathsf{P}\,p \vee (p \vee \mathsf{F}\,p))$	(t)	(1)
4.	$\neg\,\mathsf{P}\,p$	(t)	(3)
5.	$\neg(p \vee \mathsf{F}\,p)$	(t)	(3)
6.	$\neg p$	(t)	(5)
7.	$\neg\,\mathsf{F}\,p$	(t)	(5)
8.	$s < t$		(2)
9.	$\mathsf{F}\,p$	(s)	(2)
10.	$s < r$		(9)
11.	p	(r)	(9)

12. $t < r$	13. $t = r$	14. $r < t$
15. $\neg p$ (r) (7,12)	16. $\neg p$ (r) (6,13)	17. $\neg p$ (r) (4,16)
×	×	×

(f)

1.	$\neg(\mathsf{F}(\mathsf{G}\,q \wedge \neg p) \to \mathsf{G}(p \to (\mathsf{G}\,p \to q)))$	(t)	(Ass.)
2.	$\mathsf{F}(\mathsf{G}\,q \wedge \neg p)$	(t)	(1)
3.	$\neg\,\mathsf{G}(p \to (\mathsf{G}\,p \to q))$	(t)	(1)
4.	$t < s$		(2)
5.	$\mathsf{G}\,q \wedge \neg p$	(s)	(2)
6.	$\mathsf{G}\,q$	(s)	(5)
7.	$\neg p$	(s)	(5)
8.	$t < r$		(3)
9.	$\neg(p \to (\mathsf{G}\,p \to q))$	(r)	(3)
10.	p	(r)	(9)
11.	$\neg(\mathsf{G}\,p \to q)$	(r)	(9)
12.	$\mathsf{G}\,p$	(r)	(11)
13.	$\neg q$	(r)	(11)

14. $s < r$	15. $s = r$	16. $r < s$
17. q (r) (6,14)	18. p (s) (10,15)	19. p (s) (12,16)
×	×	×

Exercise 7.9

(a) For example, $\mathsf{G}\,A \to \mathsf{F}\,A$.

(b) For example, $\mathsf{H}\,A \to \mathsf{P}\,A$.

(c) No schema corresponds to the class of frames with a last time. If we also assume transitivity and quasi-connected (see page 147), then $\mathsf{G}(A \wedge \neg A) \vee \mathsf{F}\,\mathsf{G}(A \wedge \neg A)$ works.

(d) No schema corresponds to the class of frames with a first time. If we also assume transitivity and quasi-connectedness, then $\mathsf{H}(A \wedge \neg A) \vee \mathsf{P}\,\mathsf{H}(A \wedge \neg A)$ works.

Exercise 7.10

Assume a frame is dense. Suppose for reductio that some instance of $\mathsf{F}\,A \to \mathsf{F}\,\mathsf{F}\,A$ is false at some point t in some model M based on that frame. Then $\mathsf{F}\,A$ is true at t and

$\mathsf{F}\,\mathsf{F}\,A$ is false. Since $\mathsf{F}\,A$ is true at t, it follows by definition 7.2 that A is true at some point s such that $t < s$. By density, there is a point r such that $t < r < s$. But since A is true at s, $\mathsf{F}\,A$ is true at r, and so $\mathsf{F}\,\mathsf{F}\,A$ is true at t; contradiction.

In the other direction, we have to show that if a frame isn't dense then some instance of $\mathsf{F}\,A \to \mathsf{F}\,\mathsf{F}\,A$ is false at some point t in some model M based on that frame. We take the simplest instance $\mathsf{F}\,p \to \mathsf{F}\,\mathsf{F}\,p$. If a frame isn't dense then there are points t, s such that $t < s$ and no point lies in between t and s. Let V be an interpretation function that makes p true at s and false everywhere else. Then $\mathsf{F}\,p$ is true at t but $\mathsf{F}\,\mathsf{F}\,p$ is false. So $\mathsf{F}\,p \to \mathsf{F}\,\mathsf{F}\,p$ is false at t.

Exercise 7.11

Without assumptions about the flow of time there is no way to express in $\mathfrak{L}_T$ that p is true at all times (or at some time). In linear flows, $p \land \mathsf{H}\,p \land \mathsf{G}\,p$ does the job.

Exercise 7.12

Suppose $\neg\,\mathsf{P}\,A \land \neg A \land \neg\,\mathsf{F}\,A$ is true at the present time t. Then $\neg\,\mathsf{P}\,\mathsf{F}\,A$ is true (at t). By (S2), we can infer $\Box\neg\,\mathsf{P}\,\mathsf{F}\,A$. But $\neg\,\mathsf{P}\,\mathsf{F}\,A$ K_t-entails $\neg(\mathsf{F}\,A \land \mathsf{P}(A \lor \neg A))$. Since the box is closed under logical consequence, this means that $\Box\neg(\mathsf{F}\,A \land \mathsf{P}(A \lor \neg A))$ is true at t. Since t is not the first time, $\mathsf{P}(A \lor \neg A)$ is true at t, and so $\Box\,\mathsf{P}(A \lor \neg A)$ is true at t as well, by (S1). have $\neg(\mathsf{F}\,A \land \mathsf{P}(A \lor \neg A))$ and $\mathsf{P}(A \lor \neg A)$ together entail have $\neg\,\mathsf{F}\,A$. Since the box is closed under logical consequence, it follows that $\Box\,\mathsf{F}\,A$ is true at t.

Exercise 7.13

Consider a model with three times ordered by $s < t$ and $s < r$. Assume p is true at t and not at r. Then $p \to \mathsf{H}\,\mathsf{F}\,p$ is false on the Peircean interpretation.

Exercise 7.14

(a)–(d) are valid, (e) is invalid.

To show that a schema is valid, assume for reductio that there is some time t on some history H in some model M at which the schema is false. Then (repeatedly) use definition 7.3 to derive a contradiction.

For (e), consider a model with three times t, s, r such that $s < t$, $r < t$, and neither

$s \prec r$ nor $r \prec s$. Let q be true at s and false at the other two times. $\mathsf{P}\, q \to \Box\, \mathsf{P}\, \Diamond q$ is false at t on the history $\langle s, t \rangle$.

Exercise 7.15

A sentence A is super-valid iff $M, t \models A$ for all temporal models M and times t in M. By supervaluationism, this holds iff $M, H, t \models A$ for all M, t, and histories H containing t. That's how Ockhamist validity was originally defined.

Exercise 7.16

Ockham-entailment is stronger than super-entailment: whenever A Ockham-entails B, then A super-entails B, but not the other way around.

Suppose A Ockham-entails B. Let t be any time in any temporal model at which A is true, i.e.: true relative to all histories through t. Since A Ockham-entails B, B is true at t relative to all histories through t. So A super-entails B.

But suppose A super-entails B. Let t be any time on any history h in any temporal model at which A is true. We can't infer that B is true at t on h, for A may be false at t relative to other histories h'. So we can't infer that A Ockham-entails B. Indeed, $\mathsf{F}\, p$ super-entails $\Box\, \mathsf{F}\, p$, but $\mathsf{F}\, p$ does not Ockham-entail $\Box\, \mathsf{F}\, p$.

Exercise 7.17

$(A \wedge \neg A)\, \mathsf{U}\, A$.

Exercise 7.18

$\mathsf{A}\, p \to p$.