Chapter ??

Exercise 6.1

(a) $\mathsf{O}\neg p$; p: You go into the garden.
(b) $\mathsf{O}\neg p$; p: You go into the garden.
(c) $\mathsf{O}\,p$; p: Jones helps his neighbours.
(d) $\mathsf{O}(p \to q)$; p: Jones helps his neighbours, q: Jones tells his neighbours that he's coming.
(e) You might try $\mathsf{O}(\neg p \to \neg q)$ or $\neg p \to \mathsf{O}\neg q$ p: Jones helps his neighbours, q: Jones tells his neighbours that he's coming.

See section **??**, especially exercise **??**, for why neither translation of (e) is fully satisfactory.

Exercise 6.2

(a): $\Box(\mathsf{N} \to (\Box(\mathsf{N} \to A) \to A))$. (b): use wolfgangschwarz.net/trees/.

Exercise 6.3

$\mathsf{P}\,A$ could be defined as $\neg\Box(\mathsf{N} \to \neg A)$, or more simply (and equivalently) as $\Diamond(\mathsf{N} \land A)$.

Exercise 6.4

Transitivity (if wRv and vRu then wRu) and euclidity (if wRv and wRu then vRu) both state that if v is ideal and u is ideal then u is ideal.

Exercise 6.5

R is euclidean if $\forall x\forall y\forall z((xRy \land xRz) \to yRz)$. Suppose wRv. Instantiating the universal formula with w for x and with v for y and z, we have $(wRv \land wRv) \to vRv$. So vRv.

Exercise 6.6

Consider the example from the text, where w is the actual world (in the UK) and u is a w-accessible world at which everyone drives on the left although the law says that one must drive on the right. A typical world accessible from u will be a world at which people drive on the right. This world will not be accessible from w. So we have a counterexample to transitivity. We also have a counterexample to euclidity because we have wRu and wRu but not uRu. (Euclidity entails shift reflexivity.)

Exercise 6.7

Use https://www.wolfgangschwarz.net/trees/. (Write O as a box and P as a diamond. For D, make the accessibility relation serial; for KD45, make it serial, transitive, and euclidean.)

Exercise 6.8

(Dual1) says that $\neg\Diamond A$ is equivalent to $\Box\neg A$. If nothing is permitted then $\neg\Diamond A$ is true for all A. But if nothing is forbidden then $\Box\neg A$ is false for all A.

(Dual2) says that $\neg\Box A$ is equivalent to $\Diamond\neg A$. If nothing is forbidden then $\neg\Box A$ is true for all A. But if nothing is permitted then $\Diamond\neg A$ is false for all A.

Exercise 6.9

In the described situation, it ought to be the case that Amy is either obligated to help Betty or obligated to help Carla, but Amy is neither obligated to help Betty nor to help Carla. So if p translates 'Amy helps Betty' and q 'Amy helps Carla', we seem to have $\mathsf{O}(\mathsf{O}\,p \vee \mathsf{O}\,q)$ and $\neg\,\mathsf{O}\,p$ and $\neg\,\mathsf{O}\,q$. But these assumptions are inconsistent in K5. You can draw a K5-tree (using the K-rules and the Euclidity rule) starting with $\mathsf{O}(\mathsf{O}\,p \vee \mathsf{O}\,q)$ and $\neg\,\mathsf{O}\,p$ and $\neg\,\mathsf{O}\,q$ on which all branches close. This shows that there is no world in any euclidean model at which the three assumptions are true.

Exercise 6.10

(a) I argue by contraposition. Suppose some sentence $\mathsf{O}\,\mathsf{O}\,A \to \mathsf{O}\,A$ is invalid on a frame. This means that at some world w in some model M based on the frame, $\mathsf{O}\,\mathsf{O}\,A$ is true while $\mathsf{O}\,A$ is false. It follows that there is a world accessible from w at which A is false and $\mathsf{O}\,A$ true. So $\mathsf{O}\,A \to A$ is false at v. So $\mathsf{O}(\mathsf{O}\,A \to A)$ is false at w. (You could also give a tree proof with the K-rules showing that (U) entails (C4).)

(b) It is not enough to give a model in which some instance of (C4) is true at some world while the corresponding instance of (U) is false. For a counterexample, you need to give a *frame* on which every instance of (C4) is valid but not every instance of (U). Here is one such frame: $W = \{w, v\}$, wRw, wRv, and vRw.

Exercise 6.11

Since we assume that there is always at least one best world among the accessible worlds, and the accessible worlds comprise just one world, it follows that $\Box A$ is true at w iff A is true at w. The logic we get is the "Triv" logic that is axiomatized by adding the (Triv)-schema $\Box A \leftrightarrow A$ to the standard axioms and rules for K. This logic is stronger than S5: all S5-valid sentences are Triv-valid. (We also have, among other things, all instances of $\Box A \leftrightarrow \Diamond A$.) The choice between absolutism and relativism makes no difference.

Exercise 6.12

One possible answer: Let w be the world in which the story takes place. At w, Amy doesn't make any promises and isn't helping anyone. Let v be a world at which Amy promises to help Betty and keeps her promise. Let u be a world at which Amy promises to help Carla and keeps her promise. v and u are better than w, and neither is better than the other. v and u are circumstantially accessible from w, but not from each other: we can easily make promises, but if we've made a promise we can't easily dispose of the commitment.

Exercise 6.13

Use wolfgangschwarz.net/trees/.

Exercise 6.14

(c) can obviously be translated as $\mathsf{O}\, p$, (f) as $\neg p$.

If (d) is translated as $p \rightarrow \mathsf{O}\, q$ then it is entailed by (f). We would violate independence. (More directly: the translation can't be right because it is easy to think of a scenario in which (f) is true but (d) false.) Assume then that (d) is translated as $\mathsf{O}(p \rightarrow q)$.

If (e) is similarly translated as $\mathsf{O}(\neg p \rightarrow \neg q)$ then it is entailed by (c). We would violate independence. (More directly: it is easy to think of a scenario in which (c) is true but (e) false.) If (e) is translated as $\neg p \rightarrow \mathsf{O}\,\neg q$ then (a)–(c) are logically inconsistent. To show this, you can, for example, start a D-tree with the four assumptions $\mathsf{O}\, p$, $\neg p$, $\mathsf{O}(p \to q)$, and $\neg p \to \mathsf{O}\, \neg q$, all at the same world w. The tree will close.

Exercise 6.15

Simply replace ‘all’ in the semantics for $\mathsf{O}(B/A)$ with ‘some’.

Exercise 6.16

Ross’s Paradox: ‘Alice must be in the office or in the library’ seems to imply that Alice might be in the office and that she might be in the library.

The Paradox of Free Choice: ‘Alice might be in the office or in the library’ seems to imply that Alice might be in the office and that she might be in the library.

Exercise 6.17

Whenever $X \in N(w)$ then all sets that have X as a subset are in $N(w)$.

Exercise 6.18

For every world w, every member of $N(w)$ contains w.

Exercise 6.19

In Kripke semantics, $\Box p$ and $\Box q$ together entail $\Box(p \wedge q)$. But if the probability of p is above the threshold and the probability of q is above the threshold, it does not follow that the probability of $p \wedge q$ is above the threshold. For example, we could have $\Pr(p) = 0.95$, $\Pr(q) = 0.94$, and $\Pr(p \wedge q) = 0.95 \times 0.94 = 0.893$.