Chapter 1

Exercise 5.1

For an agent who knows all truths only the actual world is epistemically accessible. For an agent who knows nothing all worlds are epistemically accessible.

Exercise 5.2

(a) $\mathsf{K}(r \vee s)$
r: It is raining; s: It is snowing

(b) $\mathsf{K}\,r \vee \mathsf{K}\,s$
r: It is raining; s: It is snowing

(c) $\mathsf{K}\,r \vee \mathsf{K}\,\neg r$
r: It is raining

(d) This sentence is ambiguous. On one reading, it could be translated as $\mathsf{M}\,g \rightarrow \mathsf{K}\,g$, on the other as $\mathsf{K}(\mathsf{M}\,g \rightarrow g)$
g: You are guilty

Exercise 5.3

You can use umsu.de/trees/ to create the tree proof. We can assume S5 for the box because it quantifies unrestrictedly over all worlds (as in chapter **??**).

Exercise 5.4

(NT) is valid on all and only the frames in which no world can see any world.

Exercise 5.5

We assume that ignorance of A can be formalized as $A \wedge \neg\,\mathsf{K}\,A$. Ignorance of ignorance of A is therefore formalized as $(A \wedge \neg\,\mathsf{K}\,A) \wedge \neg\,\mathsf{K}(A \wedge \neg\,\mathsf{K}\,A)$. A tree proof shows that the former K-entails the latter.

Exercise 5.6

In a Gettier case, the relevant proposition p (say, that you're looking at a barn) is true but unknown. By (0.4), it would follow that the agent knows that they don't know p. But in a typically Gettier case the agent does not know that they don't know p.

Exercise 5.7

All except (a) and (d) are correct. You can find trees or counterexamples for (a)-(e) on umsu.de/trees/ if you write K as a box and M as a diamond. Here is a tree for (f):

1.	$\neg((\mathsf{M}\,\mathsf{K}\,p \land \mathsf{M}\,\mathsf{K}\,q) \to \mathsf{M}\,\mathsf{K}(p \land q))$	(w)	(Ass.)
2.	$\mathsf{M}\,\mathsf{K}\,p \land \mathsf{M}\,\mathsf{K}\,q$	(w)	(1)
3.	$\neg\,\mathsf{M}\,\mathsf{K}(p \land q)$	(w)	(1)
4.	$\mathsf{M}\,\mathsf{K}\,p$	(w)	(2)
5.	$\mathsf{M}\,\mathsf{K}\,q$	(w)	(2)
6.	wRv		(4)
7.	$\mathsf{K}\,p$	(v)	(4)
8.	wRu		(5)
9.	$\mathsf{K}\,q$	(u)	(5)
10.	vRt		(6,8,Con)
11.	uRt		(6.8,Con)
12.	wRt		(6.10,Tr)
13.	$\neg\,\mathsf{K}(p \land q)$	(t)	(3,12)
14.	tRs		(13)
15.	$\neg(p \land q)$	(s)	(13)

16.	$\neg p$	(s)	(15)	17.	$\neg q$	(s)	(15)
18.	vRs		(10.14,Tr)	19.	uRs		(11.14,Tr)
20.	p	(s)	(7,18)	21.	q	(s)	(9,19)
	×				×		

Exercise 5.8

see https://plato.stanford.edu/entries/dynamic-epistemic/appendix-B-solutions.html (where all the dates are 10 days later than they are in my version).

Exercise 5.9

(a) and (b) are valid, (c) and (d) are invalid. Here is a tree proof for (a).

1.	$\neg(\mathsf{M}_1\,\mathsf{K}_2\,p \to \mathsf{M}_1\,p)$	(w)	(Ass.)
2.	$\mathsf{M}_1\,\mathsf{K}_2\,p$	(w)	(1)
3.	$\neg\,\mathsf{M}_1\,p$	(w)	(1)
4.	wR_1v		(2)
5.	$\mathsf{K}_2\,p$	(v)	(2)
6.	$\neg p$	(v)	(3,4)
7.	vR_2v		(Refl.)
8.	p	(v)	(5,7)
	x		

The tree for (c) doesn't close:

1.	$\neg(\mathsf{M}_1\,\mathsf{K}_2\,p \to \mathsf{M}_2\,\mathsf{K}_1\,p)$	(w)	(Ass.)
2.	$\mathsf{M}_1\,\mathsf{K}_2\,p$	(w)	(1)
3.	$\neg\,\mathsf{M}_2\,\mathsf{K}_1\,p$	(w)	(1)
4.	wR_1v		(2)
5.	$\mathsf{K}_2\,p$	(v)	(2)
6.	vR_2v		(Refl.)
7.	p	(v)	(5,6)
8.	wR_2w		(Refl.)
9.	$\neg\,\mathsf{K}_1\,p$	(w)	(3,8)
10.	wR_1u		(9)
11.	$\neg p$	(u)	(9)

We could add a few more applications of Reflexivity, but the tree would remain open. It also gives us a countermodel: let $W = \{w, v, u\}$; w has 1-access to v and u; each world has 1- and 2-access to itself; $V(p) = \{v\}$. In this model, at world w, $\mathsf{M}_1\,\mathsf{K}_2\,p$ is true while $\mathsf{M}_2\,\mathsf{K}_1\,p$ is false.

Cases (b) and (d) are similar.

Exercise 5.10

The (5)-schema for E_G states that $\neg\,\mathsf{E}_G\,\neg A \to \mathsf{E}_G\,\neg\,\mathsf{E}_G\,\neg A$. To show that some instance of this is invalid, we need to find a case where some instance of $\neg\,\mathsf{E}_G\,\neg A$ is true while $\mathsf{E}_G\,\neg\,\mathsf{E}_G\,\neg A$ is false. We can take the simplest instance, with $A = p$. Assume the relevant group has two agents, and consider a world w at which $\mathsf{K}_1\,\neg p$ and $\neg\,\mathsf{K}_2\,\neg p$ are true. By the assumption that (5) is valid for K_i, $\mathsf{K}_2\,\neg\,\mathsf{K}_2\,\neg p$ is also true at w. But $\mathsf{K}_1\,\neg\,\mathsf{K}_2\,\neg p$ can be false (at w). If it is, then $\neg\,\mathsf{E}_G\,\neg p$ is true at w while $\mathsf{E}_G\,\neg\,\mathsf{E}_G\,\neg p$ is false.

Exercise 5.11

No, a transitive, serial, and euclidean relation is not always symmetric. Counterexample: wRv, vRv. This means that not all instances of (B) (which corresponds to symmetry) are valid in KD45.

Exercise 5.12

You can e.g. do a tree proof, using B as the box.

Exercise 5.13

Let A be an arbitrary proposition.

By (PI), $\mathsf{B}\,A \to \mathsf{K}\,\mathsf{B}\,A$ is valid. By (KB), so is $\mathsf{K}\,\mathsf{B}\,A \to \mathsf{B}\,\mathsf{B}\,A$. By propositional logic, these entail $\mathsf{B}\,A \to \mathsf{B}\,\mathsf{B}\,A$.

By (NI), $\neg\,\mathsf{B}\,\neg A \to \mathsf{K}\,\neg\,\mathsf{B}\,\neg A$ is valid. By (KB), so is $\mathsf{K}\,\neg\,\mathsf{B}\,\neg A \to \mathsf{B}\,\neg\,\mathsf{B}\,\neg A$. By propositional logic, these entail $\neg\,\mathsf{B}\,\neg A \to \mathsf{B}\,\neg\,\mathsf{B}\,\neg A$.

Exercise 5.14

The left-to-right direction is (KB). For the right-to-left direction, let A be an arbitrary proposition. By (SB), $\mathsf{B}\,A \to \mathsf{B}\,\mathsf{K}\,A$ is valid. By (D) for belief, $\mathsf{B}\,\mathsf{K}\,A \to \neg\,\mathsf{B}\,\neg\,\mathsf{K}\,A$ is valid. The contraposition of (KB) gives us $\neg\,\mathsf{B}\,\neg\,\mathsf{K}\,A \to \neg\,\mathsf{K}\,\neg\,\mathsf{K}\,A$. Finally, the contraposition of (5) for knowledge yields $\neg \mathsf{K} \neg A \to \mathsf{K} A$. The target proposition $\mathsf{B} A \to \mathsf{K} A$ is a truth-functional consequence of these four propositions.

Exercise 5.15

If the logic of belief is KD45 then $\Box\Diamond p$ is equivalent to $\Diamond p$ (as you can show, for example, with a tree proof).

Exercise 5.16

Suppose $\mathsf{B}(p \land \neg \mathsf{B} p)$. In any logic that extends K, it follows that $\mathsf{B} p$ and $\mathsf{B} \neg \mathsf{B} p$. By (4), $\mathsf{B} p$ entails $\mathsf{B} \mathsf{B} p$. Now we have $\mathsf{B} \neg \mathsf{B} p$ and $\mathsf{B} \mathsf{B} p$, which violates (D).