Chapter 5

Exercise 5.1

For an agent who knows all truths only the actual world is epistemically accessible. For an agent who knows nothing all worlds are epistemically accessible.

Exercise 5.2

If some proposition B is logically entailed by $A_1, \ldots, A_n$, then there is a derivation of B from $A_1, \ldots, A_n$ in which each individual step is obvious.

Exercise 5.3

(a) $\mathsf{K}(r \lor s)$
r: It is raining; s: It is snowing

(b) $\mathsf{K}\, r \lor \mathsf{K}\, s$
r: It is raining; s: It is snowing

(c) $\mathsf{K}\, r \lor \mathsf{K}\, \neg r$
r: It is raining

(d) This sentence is ambiguous. On one reading, it could be translated as $\mathsf{M}\, g \to Kng$, on the other as $\mathsf{K}(\mathsf{M}\, g \to g)$
g: You are guilty

Exercise 5.4

You can use wolfgangschwarz.net/trees/ to see the tree proof. We can assume S5 for the box because quantifies unrestrictedly over all worlds (as in chapter 2).

Exercise 5.5

You can't find such a frame. A logic in which (NT) is valid can't be defined by putting restrictions on the accessibility relation in Kripke models.

Exercise 5.6

We assume that ignorance of A can be formalized as $A \land \neg \mathsf{K} A$. Ignorance of ignorance of A is therefore formalized as $(A \land \neg \mathsf{K} A) \land \neg \mathsf{K}(A \land \neg \mathsf{K} A)$. A tree proof shows that the former K-entails the latter.

Exercise 5.7

In a Gettier case, the relevant proposition p (say, that you're looking at a barn) is true but unknown. By (0.4), it would follow that the agent knows that they don't know p. But in a typically Gettier case the agent does not know that they don't know p.

Exercise 5.8

All except (a) and (d) are correct. You can find trees or counterexamples for (a)-(e) on wolfgangschwarz.net/trees/ if you write K as a box and M as a diamond. Here is a tree for (f):

1.	$\neg((\mathsf{MK}\,p \wedge \mathsf{MK}\,q) \rightarrow \mathsf{MK}(p \wedge q))$	(w)	(Ass.)
2.	$\mathsf{MK}\,p \wedge \mathsf{MK}\,q$	(w)	(1)
3.	$\neg\,\mathsf{MK}(p \wedge q)$	(w)	(1)
4.	$\mathsf{MK}\,p$	(w)	(2)
5.	$\mathsf{MK}\,q$	(w)	(2)
6.	wRv		(4)
7.	$\mathsf{K}\,p$	(v)	(4)
8.	wRu		(5)
9.	$\mathsf{K}\,q$	(u)	(5)
10.	vRt		(6,8,Con)
11.	uRt		(6.8,Con)
12.	wRt		(6.10,Tr)
13.	$\neg\,\mathsf{K}(p \wedge q)$	(t)	(3,12)
14.	tRs		(13)
15.	$\neg(p \wedge q)$	(s)	(13)

16.	17.	$\neg p$	$\neg q$	(s)	(15)	(s)	(15)
18.	19.	vRs	uRs		(10.14,Tr)		(11.14,Tr)
20.	21.	p	q	(s)	(7,18)	(s)	(9,19)
		x	x				

Exercise 5.9

see https://plato.stanford.edu/entries/dynamic-epistemic/appendix-B-solutions.html (where all the dates are 10 days later than they are in my version).

Exercise 5.10

(a) and (b) are valid, (c) and (d) are invalid. Here is a tree proof for (a).

1.	$\neg(\mathsf{M}_1\,\mathsf{K}_2\,p \to \mathsf{M}_1\,p)$	(w)	(Ass.)
2.	$\mathsf{M}_1\,\mathsf{K}_2\,p$	(w)	(1)
3.	$\neg\,\mathsf{M}_1\,p$	(w)	(1)
4.	wR_1v		(2)
5.	$\mathsf{K}_2\,p$	(v)	(2)
6.	$\neg p$	(v)	(3,4)
7.	vR_2v		(Refl.)
8.	p	(v)	(5,7)
	x		

The tree for (c) doesn't close:

1.	$\neg(\mathsf{M}_1\,\mathsf{K}_2\,p \to \mathsf{M}_2\,\mathsf{K}_1\,p)$	(w)	(Ass.)
2.	$\mathsf{M}_1\,\mathsf{K}_2\,p$	(w)	(1)
3.	$\neg\,\mathsf{M}_2\,\mathsf{K}_1\,p$	(w)	(1)
4.	wR_1v		(2)
5.	$\mathsf{K}_2\,p$	(v)	(2)
6.	vR_2v		(Refl.)
7.	p	(v)	(5,6)
8.	wR_2w		(Refl.)
9.	$\neg\,\mathsf{K}_1\,p$	(w)	(3,8)
10.	wR_1u		(9)
11.	$\neg p$	(u)	(9)

We could add a few more applications of Reflexivity, but the tree would remain open. It also gives us a countermodel: let $W = \{w, v, u\}$; w has 1-access to v and u; each world has 1- and 2-access to itself; $V(p, v) = 1$ and $V(p, u) = 0$. In this model, at world w, $\mathsf{M}_1\,\mathsf{K}_2\,p$ is true while $\mathsf{M}_2\,\mathsf{K}_1\,p$ is false.

Cases (b) and (d) are similar.

Exercise 5.11

The (5)-schema for E_G states that $\neg\,\mathsf{E}_G\,\neg A \rightarrow \mathsf{E}_G\,\neg\,\mathsf{E}_G\,\neg A$. To show that some instance of this is invalid, we need to find a case where some instance of $\neg\,\mathsf{E}_G\,\neg A$ is true while $\mathsf{E}_G\,\neg\,\mathsf{E}_G\,\neg A$ is false. We can take the simplest instance, with $A = p$. Assume the relevant group has two agents, and consider a world w at which $\mathsf{K}_1\,\neg p$ and $\neg\,\mathsf{K}_2\,\neg p$ are true. By the assumption that (5) is valid for K_i, $\mathsf{K}_2\,\neg\,\mathsf{K}_2\,\neg p$ is also true at w. But $\mathsf{K}_1\,\neg\,\mathsf{K}_2\,\neg p$ can be false (at w). If it is, then $\neg\,\mathsf{E}_G\,\neg p$ is true at w while $\mathsf{E}_G\,\neg\,\mathsf{E}_G\,\neg p$ is false.

Exercise 5.12

No, a transitive, serial, and euclidean relation is not always symmetric. Counterexample: wRv, vRv. This means that not all instances of (B) (which corresponds to symmetry) are valid in KD45.

Exercise 5.13

You can e.g. do a tree proof, using B as the box.

Exercise 5.14

Let A be an arbitrary proposition.

By (PI), $\mathsf{B}\,A \rightarrow \mathsf{K}\,\mathsf{B}\,A$ is valid. By (KB), so is $\mathsf{K}\,\mathsf{B}\,A \rightarrow \mathsf{B}\,\mathsf{B}\,A$. By propositional logic, these entail $\mathsf{B}\,A \rightarrow \mathsf{B}\,\mathsf{B}\,A$.

By (NI), $\neg\,\mathsf{B}\,\neg A \rightarrow \mathsf{K}\,\neg\,\mathsf{B}\,\neg A$ is valid. By (KB), so is $\mathsf{K}\,\neg\,\mathsf{B}\,\neg A \rightarrow \mathsf{B}\,\neg\,\mathsf{B}\,\neg A$. By propositional logic, these entail $\neg\,\mathsf{B}\,\neg A \rightarrow \mathsf{B}\,\neg\,\mathsf{B}\,\neg A$.

Exercise 5.15

The left-to-right direction is (KB). For the right-to-left direction, let A be an arbitrary proposition. By (SB), $\mathsf{B}\,A \rightarrow \mathsf{B}\,\mathsf{K}\,A$ is valid. By (D) for belief, $\mathsf{B}\,\mathsf{K}\,A \rightarrow \neg\,\mathsf{B}\,\neg\,\mathsf{K}\,A$ is valid. (KB) gives us $\neg\,\mathsf{B}\,\neg\,\mathsf{K}\,A \rightarrow \neg\,\mathsf{K}\,\neg\,\mathsf{K}\,A$. Finally, (5) for knowledge yields $\neg\,\mathsf{K}\,\neg A \rightarrow \mathsf{K}\,A$. The target proposition $\mathsf{B}\,A \rightarrow \mathsf{K}\,A$ is a truth-functional consequence of these four propositions.

Exercise 5.16

If the logic of belief is KD45 then $\Box\Diamond p$ is equivalent to $\Diamond p$ (as you can show, for example, with a tree proof).

Exercise 5.17

Suppose $\mathsf{B}(p \land \neg \mathsf{B}\, p)$. In any logic that extends K, it follows that $\mathsf{B}\, p$ and $\mathsf{B} \neg \mathsf{B}\, p$. By (4), $\mathsf{B}\, p$ entails $\mathsf{B}\,\mathsf{B}\, p$. Now we have $\mathsf{B} \neg \mathsf{B}\, p$ and $\mathsf{B}\,\mathsf{B}\, p$, which violates (D).