Chapter 4

Exercise 4.1

Methods A and B are genuine proof methods. Method C is not because there is no simple mechanical check of whether an arbitrary sentence that might occur on a list is true.

Exercise 4.2

Method A is complete, but not sound. Everything that's K-valid is provable with the method, but so is everything that's not K-valid.

Method B is sound, but not complete. Since every instance of $\Box(A \vee \neg A)$ is K-valid, everything that is provable with method B is K-valid. But many K-valid sentences (e.g., $p \to p$) aren't provable with method B.

If method C were a genuine proof method, its soundness and completeness would depend on what we say about the truth-value of $\mathfrak{L}_M$-sentences. Officially, $\mathfrak{L}_M$-sentences are true or false only *relative to a world in a model*. If no particular model is salient, an $\mathfrak{L}_M$-sentence is arguably neither true nor false. And if no $\mathfrak{L}_M$-sentence is true then no $\mathfrak{L}_M$-sentence is provable with method C. The method is (vacuously) sound but not complete.

Exercise 4.3

For $A \to B$: Suppose β contains a node of the form $A \to B$ (ω) and the branch is split into two, with $\neg A$ (ω) appended to one end and B (ω) to the other. Since the expanded node is a correct statement about M under f, we have $M, f(\omega) \models A \to B$. By clause (e) of definition 3.2, it follows that either $M, f(\omega) \not\models A$ or $M, f(\omega) \models B$. By clause (b), this means that either $M, f(\omega) \models \neg A$ or $M, f(\omega) \models B$. So at least one of the resulting branches also correctly describes M.

For $\neg\Diamond A$: Suppose β contains nodes of the form $\neg\Diamond A$ (ω) and $\omega R \upsilon$, and the branch is extended by adding $\neg A$ (υ). Since $\neg\Diamond A$ (ω) and $\omega R \upsilon$ are correct statement about M under f, we have $M, f(\omega) \models \neg\Diamond A$ and $f(\omega) R f(\upsilon)$. By clause (b) of definition 3.2, $M, f(\omega) \models \neg\Diamond A$ implies $M, f(\omega) \not\models \Diamond A$. By clause (h), it follows that $M, f(\upsilon) \models \neg A$. So the extended branch correctly describes M.

Exercise 4.4

Yes. The function f can map both 'w' and 'v' to w.

Exercise 4.5

A sentence is K4-valid iff it is true at all worlds in all transitive Kripke models. We only need to check that the Transitivity rule is sound, in the sense that if a branch correctly describes a transitive model M, and the branch is extended by the Transitivity rule, then the resulting branch also correctly describes M. (The Transitivity rule allows adding a node $\omega R \upsilon$ to a branch that already contains nodes $\omega R v$ and $v R \upsilon$. If these nodes correctly describe a transitive model then so does $\omega R \upsilon$.)

Exercise 4.6

For $B \to C$: If A is a conditional $B \to C$, then β contains either $\neg B$ (ω) or C (ω). By induction hypothesis, $M, \omega \models \neg B$ or $M, \omega \models \neg C$. Either way, clauses (b) and (e) of definition 3.2 imply that $M, \omega \models A$.

For $\neg\Diamond B$: If A is a negated diamond sentence $\neg\Diamond B$, then β contains a node $\neg B$ (υ) for each world variable υ for which $\omega R \upsilon$ is on β (because the tree is fully developed). By induction hypothesis, $M, \upsilon \models \neg B$, for each such υ. By definition 4.2, it follows that $M, \upsilon \models \neg B$ for all worlds υ such that $\omega R \upsilon$. By clauses (b) and (g) of definition 3.2, it follows that $M, \omega \models A$.

Exercise 4.7

We need to check that the model induced by an open branch on a fully developed K4-tree is transitive. (Suppose the model contains worlds w, v, u for which wRv and vRu. Then the Transitivity rule has been applied to the corresponding nodes on the branch, generating a node wRu. By definition 4.2, wRu holds in the induced model.)

Exercise 4.8

Suppose A is true at some world in some Kripke model. Then $\neg A$ is K-invalid. Take any regular K-tree for $\neg A$. By observation 4.1, that tree is fully developed. By the soundness theorem for K-trees, the tree has an open branch. Let M be the model induced by some such branch β. Then M is acyclical. This is because the only rules that allow adding a node $\omega R \upsilon$ to a branch of a K-tree are the rules for expanding $\Diamond A$ and $\neg\Box A$ nodes. In both cases, the rule requires that the relevant world variable υ is new on the branch. (Call this the *novelty requirement*. Now suppose the accessibility relation in M has a cycle $\omega_1 R \omega_2$, $\omega_2 R \omega_3$, ..., $\omega_{n-1} R \omega_n$, $\omega_n R \omega_1$. Each of these facts about R must correspond to a node on β. Of these nodes, the one that was added last (to β) violates the novelty requirement. So M is acyclical.

By the Completeness Lemma, the target sentence $\neg\neg A$ is true at world w in M. So A is true at w in M. So A is true at some world in some acyclical model.

Exercise 4.9

The S5 rules are not sound with respect to K-validity. For example, $\Box p \to p$ is provable with the S5 rules, but it isn't K-valid. The rules are, however, complete with respect to K-validity. This follows from the completeness of the S5 rules and the fact that every K-valid sentence is S5-valid (observation 3.1).

Exercise 4.10

Let Γ is an infinite set of $\mathfrak{L}_M$-sentences. If Γ is K-satisfiable then obviously every finite subset of Γ is satisfiable as well. For the converse direction, assume Γ is not K-satisfiable: There is no world in any Kripke model at which all members of Γ are true. Then there is no world in any Kripke model at which all members of Γ are true while $p \land \neg p$ is false. So $\Gamma \models p \land \neg p$. By the compactness theorem, it follows that there is a finite subset Γ^- for which $\Gamma^- \models p \land \neg p$. If $\Gamma^- \models p \land \neg p$ then there is no world in any Kripke model at which all members of Γ^- are true while $p \land \neg p$ is false. Since $p \land \neg p$ is false at every world in every Kripke model, it follows that there is no world in any Kripke model at which all members of Γ^- are true. This shows that if Γ is not K-satisfiable then there is a finite subset (Γ^-) of Γ that is not K-satisfiable. Conversely, if every finite subset of Γ is K-satisfiable then Γ is K-satisfiable.

Exercise 4.11

We need to show that everything that's derivable in the axiomatic calculus for S4 is true at every world in every transitive and reflexive Kripke model. From the soundness proof for K, we know that all instances of (Dual) and (K) are true at every world in every Kripke model. From observation 3.2, we know that all instances of (T) are true at every world in every reflexive Kripke model. From observation 3.3, we know that all instances of (4) are true at every world in every transitive Kripke model. So all axioms in the S4-calculus are valid in the class of transitive and reflexive Kripke frames. Since (CPL) and (Nec) preserve validity in any class of Kripke frames, it follows that everything that's derivable in the S4-calculus is valid in the class of transitive and reflexive frames.

Exercise 4.12

(a), (b), and (c) are K-consistent, (d) is not.

Exercise 4.13

We have to show that all S4-valid sentences are provable in the axiomatic calculus for S4, which extends the calculus for T by the axiom schema $\Box A \to \Box\Box A$. The argument is by contraposition: We suppose that some sentence is not S4-provable and show that it is not S4-valid.

Suppose A is not S4-provable. Then $\{\neg A\}$ is S4-consistent. It follows by Lindenbaum's Lemma that $\{\neg A\}$ is included in some maximal S4-consistent set Γ. By definition of canonical models, this set is a world in the canonical model M_{S4} for S4. Since $\neg A$ is in Γ, it follows from the Canonical Model Lemma that $M_{S4}, \Gamma \models \neg A$. So $M_{S4}, S \not\models A$.

It remains to show that M_{S4} is reflexive and transitive.

By definition, a world v in a canonical model is accessible from w iff whenever $\Box A \in w$ then $A \in v$. Since the worlds in M_{S4} are maximal S4-consistent sets of sentences, and every such set contains every instance of the (T) schema $\Box A \to A$, there is no world in M_{S4} that contains $\Box A$ but not A. So every world in M_{S4} has access to itself.

For transitivity, suppose for some worlds w, v, u in M_{S4} we have wRv and vRu. We need to show that wRu. Given how R is defined in M_{S4}, we have to show that u contains all sentences A for which w contains $\Box A$. So let A be an arbitrary sentence for which w contains $\Box A$. Since every world in M_{S4} contains every instance of $\Box A \to \Box\Box A$, we know that w also contains $\Box\Box A$. From wRv, we can infer that v

contains $\Box A$. And from vRu, we can infer that u contains A.

Exercise 4.14

(a) Method A from exercise 4.1 is sound and complete for X. (b) No set of $\mathfrak{L}_M$-sentences is X-consistent, but every Kripke model must have at least one world.