Chapter 2

Exercise 2.1

Consider a scenario in which (say) it is raining at some worlds and not raining at others. Let p express that it is raining. In this scenario, under this interpretation, $\Diamond p$ is true, because p is true at some world. But $\Box p$ is false, because p is not true at all worlds. So there are conceivable scenarios and interpretations that render $\Diamond p$ true and $\Box p$ false.

Exercise 2.2

(b), (e), and (f) are true at w_1, the others false.

Exercise 2.3

$\Diamond p \to (q \lor \Diamond\Box p)$ is true at both worlds.

Exercise 2.4

The two definitions are not equivalent, as can be seen from the fact that the definition proposed in the exercise would render $p \models \Box p$ true. Whenever p is true at every world in a model then (by definition 2.2) $\Box p$ is also true at every world in the model. Definition 2.4 renders $p \models \Box p$ false, since there are models in which p is true at some worlds and not at others.

Exercise 2.5

By definition 2.3, a sentence is valid iff it is true at every world in every model. Suppose for reductio that $\Box p \to \Diamond p$ is false at some world w in some model. By definition 2.2, $\Box p$ is then true at w and $\Diamond p$ false. But if $\Diamond p$ is false at w then (by definition 2.2) p is false at every world in the model. And then $\Box p$ isn't true at w (by definition 2.2). Contradiction.

Exercise 2.6

Suppose A is valid – true at all worlds in all models (definition 2.3). It follows that in any given model, A is true at every world. By definition 2.2, it follows that $\Box A$ is true at every world in any model.

Exercise 2.7

$p \to \Box p$ is false at world w in the model(s) given by $W = \{w, v\}, V(p) = \{w\}$.

This shows that the *truth* of p (at a world in a model) does not entail the truth of $\Box p$ (at the world in the model), even though the *validity* of p entails the validity of $\Box p$, as per the previous exercise.

Exercise 2.8

Assume $\models A \to B$. Then there is no world in any model at which A is true and B is false. So if A is true at every world in a model, then B is also true at every world in the model. It follows that $\Box A \to \Box B$ is true at every world in every model.

Exercise 2.9

(a) Target: $p \to q$

1.	$\neg(p \to q)$	(w)	(Ass.)
2.	p	(w)	(1)
3.	$\neg q$	(w)	(1)

Countermodel: $W = \{w\}, V(p) = \{w\}, V(q) = \emptyset$.

(b) Target: $p \to \Box(p \vee q)$

1.	$\neg(p \to \Box(p \vee q))$	(w)	(Ass.)
2.	p	(w)	(1)
3.	$\neg\Box(p \vee q)$	(w)	(1)
4.	$\neg(p \vee q)$	(v)	(3)
5.	$\neg p$	(v)	(4)
5.	$\neg q$	(v)	(4)

Countermodel: $W = \{w, v\}, V(p) = \{w\}, V(q) = \emptyset$.

(c) Target: $\Box p \vee \Box\neg p$

1.	$\neg(\Box p \vee \Box\neg p)$	(w)	(Ass.)
2.	$\neg\Box p$	(w)	(1)
3.	$\neg\Box\neg p$	(w)	(1)
4.	$\neg p$	(v)	(2)
5.	$\neg\neg p$	(u)	(3)
6.	p	(u)	(5)

Countermodel: $W = \{w, v, u\}, V(p) = \{u\}$.

(d) Target: $\Diamond(p \to q) \to (\Diamond p \to \Diamond q)$

1.	$\neg(\Diamond(p \to q) \to (\Diamond p \to \Diamond q))$	(w)	(Ass.)
2.	$\Diamond(p \to q)$	(w)	(1)
3.	$\neg(\Diamond p \to \Diamond q)$	(w)	(1)
4.	$\Diamond p$	(w)	(3)
5.	$\neg\Diamond q$	(w)	(3)
6.	$p \to q$	(v)	(2)
7.	p	(u)	(4)
8.	$\neg q$	(w)	(5)
9.	$\neg q$	(v)	(5)
10.	$\neg q$	(u)	(5)

11. $\neg p$ (v) (6)	12. q (v) (6)
	×

Countermodel: $W = \{w, v, u\}, V(p) = \{u\}, V(q) = \emptyset$.

(e) $\Box\Diamond p \to p$

1.	$\neg(\Box\Diamond p \rightarrow p))$	(w)	(Ass.)
2.	$\Box\Diamond p$	(w)	(1)
3.	$\neg p$	(w)	(1)
4.	$\Diamond p$	(w)	(2)
5.	p	(v)	(4)
6.	$\Diamond p$	(v)	(2)
7.	p	(u)	(6)
8.	$\Diamond p$	(u)	(2)
9.	p	(t)	(8)

The tree grows forever. The target sentence isn't valid, but the tree method only gives us an infinite countermodel. In such a case, it may be useful to read off a model from an incomplete version of the tree and manually check whether it is a genuine countermodel. The model determined by the first five nodes of the present tree is $W = \{w, v\}, V(p) = \{v\}$, and you can confirm that it is a countermodel to the target sentence.

If you read off a model from an *incomplete* tree, you can't be sure that it is a countermodel for the target sentence. You must always double-check!

Exercise 2.10

You can enter the schemas at umsu.de/trees. After entering a formula, tick the checkbox for 'universal (S5)'. Alternatively, follow these links: (K), (T), (4), (5),

Exercise 2.11

(a), (b), (c) and (e) are valid. You can find the trees at umsu.de/trees (Remember to tick the checkbox for 'universal (S5)') or by following these links: (a), (b), (c), (e).

(d) and (f) are invalid. Here is a tree for (d):

1. $\neg((\Diamond p \wedge \Diamond q) \rightarrow \Diamond(p \wedge q))$ (w) (Ass.)
2. $\Diamond p \wedge \Diamond q$ (w) (1)
3. $\neg\Diamond(p \wedge q)$ (w) (1)
4. $\Diamond p$ (w) (2)
5. $\Diamond q$ (w) (2)
6. p (v) (4)
7. q (u) (5)
8. $\neg(p \wedge q)$ (v) (3)

9. $\neg p$ (v) (8) | 10. $\neg q$ (v) (8)
×| 11. $\neg(p \wedge q)$ (u) (3)

12. $\neg p$ (u) (11) | 13. $\neg q$ (u) (11)
14. $\neg(p \wedge q)$ (w) (3) | ×

15. $\neg p$ (w) (14) | 16. $\neg q$ (w) (14)

We can choose either of the open branches to read off a countermodel. In fact, here we get the same countermodel no matter which open branch we choose: $W = \{w, v, u\}, V(p) = \{v\}, V(q) = \{u\}$.

A tree for (e) might begin like this:

1.	$\neg(\Box\Diamond p \to \Diamond\Box p)$	(w)	(Ass.)
2.	$\Box\Diamond p$	(w)	(1)
3.	$\neg\Diamond\Box p$	(w)	(1)
4.	$\Diamond p$	(w)	(2)
5.	p	(v)	(4)
6.	$\neg\Box p$	(w)	(3)
7.	$\neg p$	(u)	(6)
8.	$\Diamond p$	(v)	(2)
9.	p	(s)	(8)
10.	$\neg\Box p$	(v)	(3)
11.	$\neg p$	(t)	(10)

The tree grows forever. The model determined by the first seven nodes of the present tree is $W = \{w, v, u\}$, $V(p) = \{v\}$. It is a countermodel to the target sentence.

Exercise 2.12

By observation 1.1, $A_1, \ldots, A_n$ entail B iff $(A_1 \land \ldots \land A_n) \to B$ is valid. To show that $A_1, \ldots, A_n$ entail B we could therefore draw a tree for $(A_1 \land \ldots \land A_n) \to B$. In practice, we can save a few steps by starting the tree with multiple assumptions: one for each of the premises $A_1, \ldots, A_n$, and one for the negated conclusion $\neg B$. (All of these are assumed to be true at world w.) If the tree closes, $A_1, \ldots, A_n$ entail B.

To show that A and B are equivalent, we can draw a tree for $A \leftrightarrow B$.