Answers to the Exercises

Chapter 1

Exercise 1.1

(a), (c), and (d) are $\mathfrak{L}_M$-sentences, (b), (e), and (f) are not.

Exercise 1.2

An operator O is truth-functional if you can figure out the truth-value of Op from the truth-value of p.

(c) and (g) are truth-functional; (a), (b), (d), and (e) are not truth-functional.

(f) is truth-functional if God is omniscient (and infallible); it is also truth-functional if God doesn't exist, or if God believes all and only false things; otherwise (f) is not truth-functional.

Exercise 1.3

In most situations, there are false propositions P for which 'it might be that P' is true (because we are not omniscient) and other false propositions Q for which 'it might be that Q' is false (because there are at least some truths we actually know).

Exercise 1.4

(a) $\Diamond p$ $\quad p$: I offended the principal.
(b) $\neg\Diamond p$ $\quad p$: It is raining.
(c) $\Diamond p$ $\quad p$: There is life on Mars.
(d) $\Diamond(p \land q)$ $\quad p$: The lights are on; q: Ada is in her office.
(e) $\Box(p \to q)$ $\quad p$: The murderer escaped through the window; q: There are traces on the ground.

Exercise 1.5

(a) $\Box p$ p: I go home.
(b) $\neg\Box p$ p: You come.
(c) $\neg\Diamond p$ p: You have another beer.
(d) $\Box(\neg p \to q)$ p: You have a ticket; q: You pay a fine.

Exercise 1.6

(a) $\Diamond p$ p: I study architecture.
(b) $\Diamond p$ p: The bridge collapses.
(c) $\neg\Diamond(p \wedge q)$ p: You are talking to me from the kitchen; q: I hear you.
(d) $\neg\Diamond p \to \neg\Diamond q$ p: You go to the station; q: You take the train.

Exercise 1.7

The following pairs are duals: (a) and (c), (b) and (d), (e) and (g), (f) and (h), (i) and (k), (l) and (l), (m) and (m).

Exercise 1.8

(b) and (e) are equivalent to $\Diamond\Diamond\neg p$, (a), (c), and (d) are not.

As a rule, you can always replace a modal operator by its dual, insert a negation on both sides, and remove any double negations to get an equivalent sentence.

Exercise 1.9

(b) and (d)

Exercise 1.10

(a) $\Diamond\Diamond A \to \Diamond A$, (b) $\Diamond\Box A \to \Box A$, (c) $\Box A \to \Diamond A$.

Exercise 1.11

(a) $\neg\Box p \wedge \neg\Box\neg p$; (b) $\Diamond p \wedge \Diamond\neg p$; (c) $\neg\nabla p \wedge p$. The last answer assumes that every necessary proposition is true. Without that assumption there is no answer to (c).

Exercise 1.12

The proposed definition is equivalent to definition 1.2 for some languages but not for others. Consider the sentence $\exists x \exists y \neg (x = y)$ in the language of predicate logic. If we treat the identity symbol as logical, this sentence contains no non-logical expressions. And the sentence is true, because there is in fact more than one object. So the sentence is true under any interpretation of its non-logical vocabulary. But it's not logically true; it doesn't logically follow from any premises whatsoever. The sentence is false in any scenario in which there is only one object.

Exercise 1.13

(a) All of them.
(b) Only (K) and (CPL).
(c) All except (T).
(d) All of them.

Exercise 1.14

(a)

1.	$\Box p \to p$	(T)
2.	$\Box(\Box p \to p)$	(1, Nec)

(b)

1.	$p \to (q \to (p \land q))$	(CPL)
2.	$\Box(p \to (q \to (p \land q)))$	(1, Nec)
3.	$\Box(p \to (q \to (p \land q))) \to (\Box p \to \Box(q \to (p \land q)))$	(K)
4.	$\Box p \to \Box(q \to (p \land q)))$	(2, 3, CPL)
5.	$\Box(q \to (p \land q))) \to (\Box q \to \Box(p \land q))$	(K)
6.	$\Box p \to (\Box q \to \Box(p \land q))$	(4, 5, CPL)
7.	$(\Box q \land \Box q) \to \Box(p \land q)$	(6, CPL)

(c)

1.	$\neg\Diamond\neg p \leftrightarrow \Box\neg\neg p$	(Dual)
2.	$\neg\neg\Diamond\neg p \leftrightarrow \neg\Box\neg\neg p$	(1, CPL)
3.	$\Diamond\neg p \leftrightarrow \neg\Box\neg\neg p$	(2, CPL)
4.	$\neg\neg p \rightarrow p$	(CPL)
5.	$\Box(\neg\neg p \rightarrow p)$	(4, Nec)
6.	$\Box(\neg\neg p \rightarrow p) \rightarrow (\Box\neg\neg p \rightarrow \Box p)$	(K)
7.	$\Box\neg\neg p \rightarrow \Box p$	(5, 6, CPL)
8.	$p \rightarrow \neg\neg p$	(CPL)
9.	$\Box(p \rightarrow \neg\neg p)$	(8, Nec)
10.	$\Box(p \rightarrow \neg\neg p) \rightarrow (\Box p \rightarrow \Box\neg\neg p)$	(K)
11.	$\Box p \rightarrow \Box\neg\neg p$	(9, 10, CPL)
12.	$\Box\neg\neg p \leftrightarrow \Box p$	(7, 11, CPL)
13.	$\neg\Box\neg\neg p \leftrightarrow \neg\Box p$	(12, CPL)
14.	$\Diamond\neg p \leftrightarrow \neg\Box p$	(3, 13, CPL)

Exercise 1.15

In an axiomatic calculus, every line in a proof is either an axiom or follows from an earlier line by one of the rules. (Nec) therefore assumes that whenever a sentence A is *provable in the axiomatic calculus*, then it is necessarily true (reading the box as ‘it is necessary that’).

The rules of the axiomatic calculus cannot be used to directly derive assumptions from arbitrary premises. To show that A entails B, you have to prove $A \rightarrow B$.